PLANE GEOMETRY

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO · DALLAS
ATLANTA · SAN FRANCISCO

MACMILLAN & CO., LIMITED
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN CO. OF CANADA, LTD.
TORONTO

o

SCHULTZE AND SEVENOAK'S

PLANE GEOMETRY

REVISED BY

ARTHUR SCHULTZE

New York
THE MACMILLAN COMPANY
LONDON: MACMILLAN & CO., LTD.
1913

Set up and electrotyped. Published September, 1913.

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE TO THE FIRST EDITION

It is generally conceded that the final aim of mathematical teaching should be not only the acquisition of practical knowledge, but that training of the student's mind which gives a distinct gain in mental power. In recognition of this principle nearly all college entrance examinations in geometry require some original work, and most text-books devote considerable space to exercises. Comparatively little, however, has been done to introduce the student *systematically* to original geometrical work. No teacher of physics or chemistry would ask a student to discover a law without so guiding his work as to enable him to reach the desired result; many text-books and teachers expect the pupil to invent geometrical proofs and to solve problems, entirely new to him, without offering any assistance further than a knowledge of the well-established theorems of all text-books. Some writers give a description of the analysis of propositions, which is entirely logical and of great advantage to a person of some mathematical knowledge, but which is usually too abstract to be of any practical value to the beginner. In this book the attempt is made to introduce the student systematically to the solution of geometrical exercises. In the beginning the exercises given in a certain group are of similar kind and related to the preceding proposition; later some general principles are developed which are of fundamental importance for original work, as, for example, the method of proving the equality of lines by means of equal triangles; the method of proving the proportionality of lines by means of similar triangles, etc.; and finally the analyses of theorems and problems are introduced, but in a more concrete form than usual.

The propositions are arranged with the view to obtaining a perfect logical and pedagogical order. An unusually large number of exercises is given, selected with care for the purpose of securing increased mental power.

The general plan and the preparation of the greater part of the book are the work of Dr. Schultze, while that of Dr. Sevenoak has been chiefly editorial.

PREFACE TO THE REVISED EDITION

The main purpose of the revision of this book has been to emphasize still further and to elaborate in greater detail the principal aim of the original edition, viz., *to introduce the student systematically to original geometric work.* To make the teaching of geometry both disciplinary and informational; to give to the student mental training instead of teaching him mere facts; to develop his power instead of making him memorize, — these are the fundamental aims of this book.

The means employed for this purpose are similar to those used in the first edition. Still greater emphasis, however, has been placed upon the general methods which may be used for the solution of original exercises. The grading and the selection of exercises have been carefully revised. All originals that appeared unfit or too difficult have been eliminated or replaced by simpler and better ones. Topics of fundamental importance, *e.g.* the methods of demonstrating the equality of lines, are represented in greater detail and illustrated by a greater number of exercises than in the first edition.

In addition to these fundamental tendencies, a number of minor improvements have been introduced, among which may be mentioned:

Improved presentation of the regular propositions. Many proofs have been simplified, a more pedagogic sequence of the propositions of Book I has been adopted, Books VI and VII have been considerably simplified, and a number of difficult theorems of minor importance have been omitted or placed in the appendix.

Simplification of the so-called "incommensurable case." As this is a claim that is made by most text-books, it may be received with some degree of skepticism, but a repeated trial of this new method will reveal its simplicity. For the more conservative teacher, however, who dislikes fundamental changes, the time-honored method is given in the appendix.

The introduction of many applied problems. These problems have been selected and arranged so as to increase the interest of the student, without sacrificing in the least the disciplinary value of the subject. Many such problems are given in the appendix.

The arrangement of the propositions and the terminology are in accord with the best modern usage. Thus, statements and reasons have been separated and placed in parallel vertical columns; the term "congruent" and the corresponding symbol are introduced and applied consistently, etc.

Many of the diagrams have been improved. The construction lines are drawn completely for most problems, graphical modes are employed for pointing out important facts, and many diagrams have been otherwise improved.

Thanks are due to Dr. J. Kahn and Mr. W. S. Schlanch for assistance in reading the proof and for helpful suggestions.

A. S.

August 1, 1913.

CONTENTS

PLANE GEOMETRY

SYMBOLS AND ABBREVIATIONS

$+$	plus, or added to.
$-$	minus, or diminished by.
$=$	equals, or is equivalent to.
$\cong$	congruent.
$\neq$	is not equal to.
$>$	is greater than.
$<$	is less than.
$\therefore$	therefore, or hence.
$\perp$	perpendicular, or is perpendicular to.
$\perp$s	perpendiculars.
$\parallel$	parallel, or is parallel to.
$\parallel$s	parallels.
$\sim$	is similar to, or similar.
$\angle$	angle.
$\angle$s	angles.
$\triangle$	triangle.
$\triangle$s	triangles.
▱	parallelogram.
▱s	parallelograms.
$\odot$	circle.
$\odot$s	circles.
$\frown$	arc; as $\overset{\frown}{AB}$, arc AB.
adj.	adjacent.

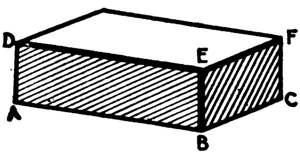
alt.	alternate.
ax.	axiom.
circum.	circumference.
comp.	complement.
con.	construction.
cor.	corollary.
corr.	corresponding.
def.	definition.
ex.	exercise.
ext.	exterior.
hom.	homologous.
hy.	hypotenuse.
hyp.	hypothesis.
iden.	identity.
int.	interior.
isos.	isosceles.
rt.	right.
st.	straight.
sub.	substitution.
sup.	supplementary, or supplement.
Q. E. D.	quod erat demonstrandum (which was to be proved).

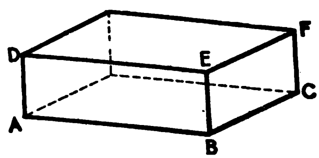
PLANE GEOMETRY

INTRODUCTION

DEFINITIONS

1. A physical body, such as a block of wood or iron, occupies a definite portion of space. The portion of space occupied by a physical body is called a *geometric solid* or a *solid.*





2. DEF. **A solid** is a limited portion of space. It has three dimensions, *length, breadth,* and *thickness.*

3. DEF. **Surfaces** are the boundaries of solids; as *ABED* or *BEFC.* They have two dimensions, *length* and *breadth.*

The boundary between a window pane and the air is a surface. Obviously such a boundary has no thickness.

4. DEF. **Lines** are the boundaries of surfaces, as *AB, AD.* (Figure of § 1.) Lines have but one dimension, *length.*

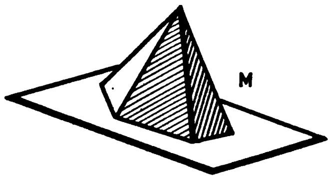
Thus, the annexed black line *AB* is not a geometric line, for it has breadth.

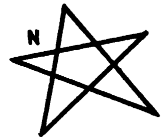
A true geometric line, however, is represented by the boundary between the black and the white.

5. DEF. **Points** are the boundaries or the extremities of lines. They are without dimensions, having position only.

Surfaces may be conceived as existing independent of the solids whose boundaries they form. In like manner, lines and points may exist independently in space.

6. Def. A **geometric figure** is a point, line, surface, or solid, or any combination of any or all of these; as M or N.

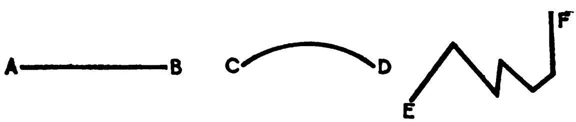




A *rectilinear figure* is a figure composed of straight lines only.

7. Def. **Geometry** is the science that treats of the properties of geometric figures.

8. The simplest line is a **straight line.** It is represented approximately by a string stretched taut between two points; as AB. The word "**line**" is frequently used to denote a straight line.



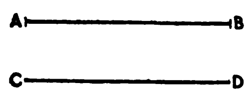
A very simple term is, as a rule, not easy to define on account of the difficulty of finding still simpler terms by which to define it. The notion of a straight line is such a simple and fundamental one that it is practically impossible to give a good definition of it.

9. Def. A **curved line** or **curve** is a line no portion of which is straight; as CD.

10. Def. A **broken line** is a line composed of different successive straight lines; as EF.

11. The expression, straight line, is used to denote both an unlimited straight line and a part of such line.

A *line of definite length*, also called a **segment**, or **line-segment**, is represented by a line whose ends are marked; as AB. The length of this line is also called the **distance** from A to B.



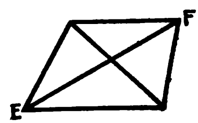
A line whose ends are not marked represents a line of indefinite length; as *CD*.

12. The **direction** of the line *AB* means the direction from *A* toward *B*; of *BA*, the direction from *B* toward *A*.

13. To **produce** the line *AB* means to prolong it through *B*; to **produce** *BA* means to prolong it through *A*.

14. Def. A **plane surface** or a **plane** is a surface such that a straight line joining any two of its points lies entirely in the surface.

15. Def. A **plane figure** is a geometric figure, all of whose points lie in the same plane; as *EF*.



16. Def. **Plane Geometry** treats of plane figures only.

17. Def. **Solid Geometry** treats of figures which are not plane.

18. When one figure can be placed upon another so that each point of one lies upon some point of the other, the figures are said to **coincide**.

19. Def. **Congruent figures** are those that can be made to coincide.

For reasons that will appear later congruent lines are frequently called *equal lines*. Similarly angles that can be superposed are usually called *equal angles*.

20. **Proof by superposition** is the method of proving the congruence of two figures by making them coincide.

21. To **bisect** a line means to divide it into two equal parts.

Thus, *AC* is bisected if $AD = DC$.

Ex. 1. What is the path of a moving point?

Ex. 2. What geometric figure is, in general, generated by a moving line? by a moving surface?

Ex. 3. Can a straight line move so that its path is not a surface?

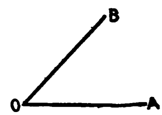
Ex. 4. How does a stone cutter use the straight edge to determine whether a surface is plane?

Ex. 5. What kind of surface is represented by each wall of a room?

Ex. 6. What kind of surface is represented by a gas-pipe?

ANGLES

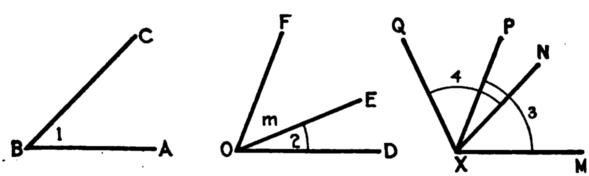
22. If a straight line *OA* revolves about one of its points *O* until the line reaches the position *OB*, then the amount of this rotation is called "*the angle AOB.*" Obviously the amount of rotation, and hence the angle, does not depend upon the length of the line which rotates.



The lines *OA* and *OB* are called the **sides** and the point *O* the **vertex** of the angle *AOB*.

The student should note that the preceding statement is not a definition, but merely an explanation of the term angle. No definition of this term exists that is free from objections.*

23. Notation. If three letters are used to denote an angle, the vertex letter should be read between the others; as angle *ABC*, angle *EOF*. A single letter at the vertex denotes the



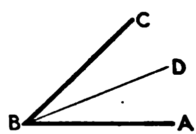
* The definition that is at present most widely used is the following one: "An angle is the figure formed by two straight lines diverging from a point." This definition, however, not only fails to explain which part of the figure really constitutes the angle, but it also makes use of the undefined term 'diverge.' Moreover it is not applicable to angles greater than 180°.

largest angle at this vertex (if there be several at this point). Thus, angle *DOF* may be read "angle *O*," angle *ABC* may be read "angle *B*."

Frequently an angle is also designated by a number, or italic letter, placed within it, as angle 1, angle 2, angle *m*.

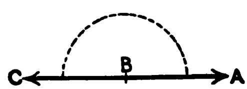
Often a curve is drawn to point out more clearly which angle is meant; as angle 2, and angle 3. An arc placed close to a number shows which angle is designated. Thus, angle *MXP* may be read "angle 3," and angle *NXQ* may be read "angle 4."

24. To **bisect** an angle means to divide it into two equal parts.



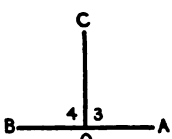
Thus, *BD* bisects angle *ABC*, if angle *ABD* = angle *DBC*. *BD* is called the **bisector** of angle *B*.

25. Def. A **straight angle** is an angle whose sides lie in the same straight line but extend in opposite directions, as *ABC*.

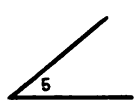


26. Def. A **right angle** is an angle equal to one half of a straight angle.

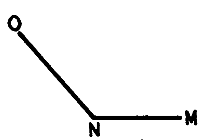
Thus, if *OC* bisects the straight angle *AOB*, angle 3 and angle 4 are right angles



RIGHT ANGLES



ACUTE ANGLE

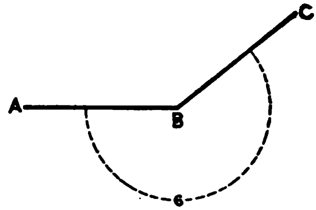


OBTUSE ANGLE

27. Def. An **acute angle** is an angle less than a right angle; as angle 5.

28. Def. An **obtuse angle** is an angle greater than a right angle, but less than a straight angle; as angle *MNO*.

29. Def. A **reflex angle** is an angle greater than a straight angle, but less than two straight angles; as angle 6.

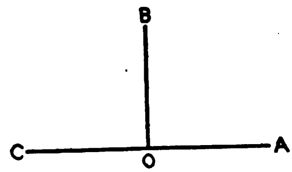


An angle denoted by the usual methods does *not* signify a reflex angle, unless designated as 'reflex angle' or indicated by an arc.

30. Def. Acute, obtuse, and reflex angles are called **oblique angles.**

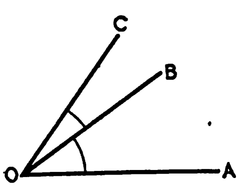
31. Def. Two lines are **perpendicular** to each other if they meet at right angles; as AC and BO.

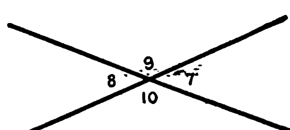
The point of meeting (O) is the **foot of the perpendicular.**



32. An angle is **measured** by finding how many times it contains a certain unit. The usual unit is the *degree*, or one-ninetieth ($\frac{1}{90}$) of a right angle. A degree is divided into sixty equal parts called *minutes*, and a minute into sixty equal parts called *seconds*. Degrees, minutes, and seconds are expressed by symbols, as $6^\circ\ 50'\ 12''$. Read *six degrees, fifty minutes, and twelve seconds.*

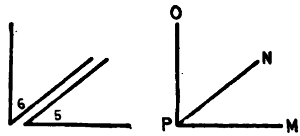
33. Def. **Adjacent angles** are two angles that have a common vertex, and a common side between them; as angles AOB and BOC.





34. Def. Two angles are **vertical angles** if the sides of each are prolongations of the sides of the other; as angles 7 and 8, or angles 9 and 10.

35. Def. Two angles are **complementary** if their sum equals a right angle.



Each is then called the *complement* of the other. Angles 5 and 6, or MPN and NPO, are complementary.

36. Def. Two angles are **supplementary** if their sum equals a straight angle.

Each angle is then called the supplement of the other. Angles 1 and 2, or angles 3 and 4, are supplementary.

Ex. 7. How many degrees are in a right angle? In a straight angle? In one half a right angle?

Ex. 8. What is the angle made by the two hands of a clock at three o'clock? At six o'clock? At two o'clock? At five o'clock?

Ex. 9. What is the angle made by the hands of a clock at 1 P.M.? At 2 : 30 P.M.? At 5 : 30 P.M.?

Ex. 10. Over an angle of how many degrees does a spoke of a wheel sweep when the wheel makes $\frac{1}{4}$ of a revolution? $\frac{1}{8}$ of a revolution? 2 revolutions?

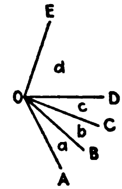
Ex. 11. How large is each angle at the center if a pie is divided into 5 equal parts? 6 equal parts?

Ex. 12. What angle is formed by lines drawn towards north and northeast? Towards S. and S.E.? Towards N.W. and S.W.?

Ex. 13. Over what angle does the large hand of a watch sweep in 10 min.? 15 min.? 30 min.? 45 min.? 1 hr.?

Ex. 14. In the diagram of Ex. 15 read by three letters: $\angle a$, $\angle b$, $\angle d$, $\angle (a + b)$.

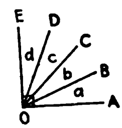
Ex. 15. In diagrams similar to the one given here find the numerical values of the required angles:



(*a*) If $\angle a = 30°$, and $\angle b = 40°$, find $\angle AOC$.
(*b*) If $\angle b = 35°$, and $\angle c = 10°$, find $\angle BOD$.
(*c*) If $\angle b = 40°$, $\angle c = 10°$, and $\angle d = 50°$, find $\angle BOE$.
(*d*) If $\angle AOC = 60°$, and $\angle b = 40°$, find $\angle a$.
(*e*) If $\angle AOD = 90°$, $\angle a = 35°$, and $\angle c = 10°$, find $\angle b$.
(*f*) If $\angle AOE = 110°$, $\angle a = 20°$, and $\angle d = 30°$, find $\angle BOD$.
(*g*) If $\angle AOC = 60°$, and $\angle a = \angle b$, find $\angle a$.
(*h*) If $\angle AOD = 75°$, and $\angle a = \angle b = \angle c$, find $\angle c$.

Ex. 16. In the preceding diagram, which angles are adjacent to $\angle BOC$? to $\angle COD$? to $\angle BOD$?

Ex. 17. In diagrams similar to the one shown, if $\angle O = 90°$:

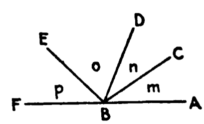


(*a*) Which angle is the complement of $\angle a$?
(*b*) Which angle is the complement of $\angle AOC$?
(*c*) Which angle is the complement of $\angle BOE$?
(*d*) If $\angle d = 20°$, find $\angle AOD$.
(*e*) If $\angle b = 20°$, and $\angle COE = 55°$, find $\angle a$.
(*f*) If $\angle AOC = 55°$, and $\angle d = 15°$, find $\angle c$.
(*g*) If $\angle a = \angle b = \angle c = \angle d$, find $\angle a$.

Ex. 18. How many degrees are in the complement of 30°? Of 35°? Of $\frac{2}{3}$ right angles? Of $n°$? Of $\frac{1}{n}$ of a right angle? Of $(10 + x)°$?

Ex. 19. How many degrees are there in an angle that is twice its complement?

Ex. 20. In diagrams similar to the annexed one, if *FBA* is a straight line,



(*a*) Which angle is the supplement of $\angle p$?
(*b*) Which angle is the supplement of $\angle DBF$?
(*c*) Which angle is the supplement of $\angle ABE$?
(*d*) If $\angle p = 40°$, find $\angle ABE$.
(*e*) If $\angle m = 30°$, and $\angle p = 35°$, find $\angle CBE$.
(*f*) If $\angle DBF = 100°$, and $\angle m = \angle n$, find $\angle m$.
(*g*) If $\angle p = 30°$, and $\angle m = \angle n = \angle o$, find $\angle o$.
* (*h*) If $\angle FBC = 140°$, and $\angle ABD = 80°$, find $\angle n$.
* (*i*) If $\angle ABD = 80°$, $\angle n = 35°$, and $\angle CBE = 85°$, find $\angle p$.

Ex. 21. How many degrees are in the supplement of 20°? of 140°? of $\frac{3}{4}$ straight angles? of n degrees? of $(50 - 3x)°$?

* **Exercises denoted by (*) are difficult.**

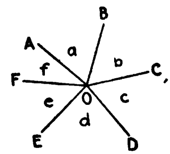
Ex. 22. How many degrees are in an angle which is three times its supplement?

Ex. 23. What kind of angle is less than its supplement? equal to its supplement? greater than its supplement?

Ex. 24. Write in algebraic symbols:

(a) The complement of n°. (b) Three times the complement of x°. (c) The supplement of $(2x)^\circ$. (d) Six times one supplement of n°.

Ex. 25. In diagrams similar to the annexed one find the numerical values of the required angles:



(a) If $\angle a = 80^\circ$, $\angle b = 50^\circ$, $\angle c = 60^\circ$, $\angle d = 90^\circ$, and $\angle e = 50^\circ$, find $\angle f$.

(b) If $\angle a = 85^\circ$, and $\angle b = 55^\circ$, find reflex angle AOC.

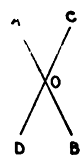
(c) If $\angle a = 85^\circ$, and $\angle AOE = 85^\circ$, find reflex angle BOE.

(d) If reflex angle $AOC = 230^\circ$, and $\angle b = 50^\circ$, find $\angle a$.

(e) If $\angle a = \angle b = \angle c = \angle d = \angle e = \angle f$, find $\angle f$.

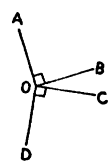
*(f) If $\angle AOC = 130^\circ$, $\angle b = 50^\circ$, $\angle BOD = 110^\circ$, and $\angle DOF = 140^\circ$, find $\angle f$.

(g) If $\angle d = 90^\circ$, and $\angle c = \angle b = \angle a = \angle f = \angle e$, find $\angle a$.



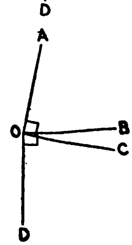
Ex. 26. If two lines, AB and CD, intersect in O, making $AOC = 60^\circ$, find the other angles.

Ex. 27. In the same diagram, if $AOC = m$ degrees, how many degrees are in DOB? in BOC?



Ex. 28. If in the annexed diagram $\angle AOB = \angle COD = 90^\circ$, find $\angle AOD$ (a) if $\angle BOC = 60^\circ$, (b) if $\angle BOC = m^\circ$.

Ex. 29. What relation exists between the angles BOC and AOD in the preceding exercise?

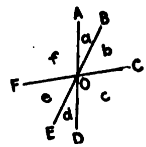


Ex. 30. If, in the annexed diagram, AO is perpendicular to CO, and BO is perpendicular to DO, find AOD, (a) if $COB = 40^\circ$, (b) if $COB = m^\circ$.

Ex. 31. What relation exists between AOD and BOC in the preceding exercise?

* **Ex. 32.** If, in the same diagram, $\angle AOC = \angle BOD = 90^\circ$, and $\angle AOD = 3\,(BOC)$, find $\angle BOC$.

Ex. 33. Three lines meet in O forming the six angles: a, b, c, d, e, and f.



(a) If $\angle a = 20°$, and $\angle b = 60°$, find $\angle c$.

* (b) If $\angle a = 15°$, and $\angle c = 95°$, find $\angle e$.

* (c) If $\angle f = 100°$, and $\angle d = 20°$, find $\angle b$.

* (d) If $\angle AOC = 85°$, and $\angle BOD = 155°$, find $\angle e$.

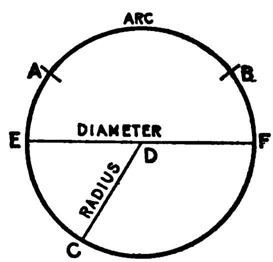
Ex. 34. Find the angle formed by the bisectors of the supplementary adjacent angles AOB and BOC, (a) if $\angle AOB = 40°$, (b) if $\angle AOB = 60°$, (c) if $\angle AOB = m°$.

Ex. 35. What is the angle formed by the bisectors of any two supplementary adjacent angles?

Ex. 36. Find the angle formed by the bisectors of the complementary adjacent angles AOB and BOC. (a) if $\angle AOB = 20°$, (b) if $\angle AOB = 30°$, (c) if $\angle AOB = m°$.

Ex. 37. What is the angle formed by the bisectors of any two complementary adjacent angles?

37. Def. A circle is a plane closed curve, all of whose points are equally distant from a fixed point; as ABC.†



The *center* (D) is the fixed point. A *radius* is any straight line from the center to a point in the circle; as DC. An *arc* is any portion of the circle; as AB. The length of the circle is called the *circumference*.

The term "circumference" is frequently used to denote the curved line, and the term "circle" to denote the area. But modern usage is against this terminology.

38. Instruments used in Geometry. Only two instruments are permitted to be used in plane geometry, viz. the *compasses* and the *ruler* or *straight edge*.

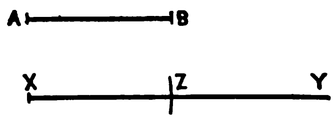
The ruler is employed for the drawing of straight lines, the compasses for the drawing of circles or arcs and the transfer of lines of definite length (segments) from one position to another.

† A line in a plane is said to be closed if it separates a finite portion of the plane from the remaining portion.

EXERCISES IN GEOMETRIC DRAWING

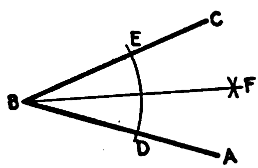
39. The following exercises are designed to familiarize the student with the use of the instruments and to impress upon him the meaning of the fundamental notions of geometry. They form, however, no logical part of plane geometry, and may be omitted without affecting the course. They are based upon three constructions, for which no proofs are given here. (Compare §§ 82 and 84.)

I. *To lay off on XY a line equal to AB.*



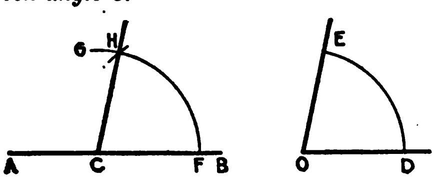
From X as a center with a radius equal to AB draw an arc intersecting XY in Z; XZ is the required line.

II. *To bisect a given angle ABC.*



From B as a center with any radius draw an arc meeting AB in D, and AC in E. From D and E as centers with a radius sufficiently large draw two arcs intersecting in F. Then the line BF is the required bisector.

III. *At a point C in a given straight line AB, to draw an angle equal to a given angle O.*



From O as a center, with any radius, describe an arc cutting the sides of $\angle O$ in D and E. From C as a center with the same radius describe arc FG, intersecting AB in F. From F as a center with a radius equal to DE,* draw an arc intersecting FG in H. Draw CH. Then $\angle BCH$ is the required angle.

Ex. 38. Draw two points A and B (indicated by small dots or crosses) and construct a line through A and B.

Ex. 39. Draw two points A and B, and construct a circle from A as a center with a radius equal to AB.

Ex. 40. Draw three points A, B, and C, and construct a straight line through each two of them.

Ex. 41. Draw two lines of definite length, AB and CD, AB being the greater one, and construct lines equal to:

(a) $AB + CD$
(b) $AB - CD$
(c) $2(AB)$
(d) $AB + 2(CD)$
(e) $3(AB) - 2(CD)$

Ex. 42. Draw an acute angle and bisect it.

Ex. 43. Draw an obtuse angle and bisect it.

Ex. 44. Draw a reflex angle and bisect it.

Ex. 45. Draw a straight angle and bisect it.

Ex. 46. Construct a right angle.

Ex. 47. At a given point C in AB draw a perpendicular to AB.

Ex. 48. Divide a given angle into 4 equal parts.

Ex. 49. Divide a given angle into 8 equal parts.

Ex. 50. Construct an angle of (a) $90°$ (b) $45°$.

Ex. 51. Construct an angle of (a) $22° 30'$ (b) $135°$.

Ex. 52. Construct an angle of (a) $270°$ (b) $67° 30'$.

Ex. 53. Construct the supplement of a given angle A.

Ex. 54. Construct one half the supplement of a given angle A.

Ex. 55. Construct the complement of a given acute angle.

Ex. 56. Construct one half the complement of a given acute angle.

* Two letters, *e.g.* DE, used as above, denote a straight line.

Ex. 57. Construct the supplement of the complement of a given acute angle.

Ex. 58. Construct the complement of the supplement of a given obtuse angle.

Ex. 59. Draw two angles A and B, A being the greater one, and construct an angle equal to

(a) $A + B$.

(b) $2A$.

(c) $180° - A$.

(d) $90° + A$.

(e) $A - B$.

(f) $\frac{A}{2}$.

(g) $\frac{A}{2} + B$.

(h) $\frac{A+B}{2}$.

(i) $\frac{A}{2} + \frac{B}{2}$.

(k) $\frac{A}{2} - \frac{B}{2}$.

(l) $90 + \frac{B}{2}$.

(m) The complement of $\frac{B}{4}$.

GENERAL TERMS

40. Def. A **theorem** is a statement the truth of which is to be demonstrated.

Thus the statement: "*If two sides of a triangle are equal, the angles opposite are equal,*" is a theorem. The conditional part of the statement is called the *hypothesis*, the assertion that is to be proved is called the *conclusion*. Thus in the above example, the hypothesis is: "*If two sides of a triangle are equal,*" and the conclusion is: "*the angles opposite are equal.*" The hypothesis is, however, not always explicitly stated, but is sometimes merely implied.

41. Def. A **problem**, in general, is a question to be solved.

The problems of geometry are solved either by constructions or by computations. Thus the problem: "To construct an angle of 45°" (Ex. 50, p. 12) is a *problem of construction.* "To compute the angle formed by the bisectors of a pair of complementary adjacent angles" (Ex. 36, p. 10) is a *problem of computation.*

42. Def. A **proposition** is a general term for a theorem or a problem.

43. Def. An **axiom** is a statement whose truth is assumed.

Thus, the statement: "If equals be added to equals the sums are equal," is an axiom.

44. Def. A **postulate** is a purely geometric axiom.

Formerly elementary textbooks restricted the term postulate to "geometric constructions whose possibility is admitted without further demonstration," but modern usage favors the more general definition given above.

45. Def. A **corollary** is a theorem easily derived from another theorem.

AXIOMS AND POSTULATES

1. *Things equal to the same thing, or to equal things, are equal to each other.*
2. *If equals are added to equals, the sums are equal.*
3. *If equals are subtracted from equals, the remainders are equal.*
4. *If equals are added to unequals, the sums are unequal in the same sense.*
5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*
6. *If unequals are subtracted from equals, the remainders are unequal in the opposite sense.*
7. *If equals are multiplied by equals the products are equal.* (Important special case: *doubles of equals are equal.*)
8. *If equals are divided by equals, the quotients are equal.* (Important special case: *halves of equals are equal.*)
9. *If one of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*
10. *The whole is equal to the sum of all its parts.*
11. *The whole is greater than any of its parts.*
12. *A quantity may be substituted for an equal one in an equation or in an inequality.* (Briefly called "*substitution.*")

13. *One straight line and only one can be drawn through two points.*

14. *A straight line is the shortest line between two points.*

15. *All straight angles are equal.*

(For Axiom 16, see p. 43 on parallel lines.) The preceding axioms and postulates are the most important ones, but others are assumed in elementary geometry. Among them may be mentioned:

A straight line may be produced to any required length.

A circle can be drawn from any point as a center with any line as a radius.

A geometric figure may be moved from one position to another without change of form or size, etc.

The first twelve axioms are general axioms, the following ones are geometric axioms or postulates.

From the thirteenth axiom these truths obviously follow:

(*a*) Two points determine a straight line.

(*b*) Two straight lines can intersect in only one point.

(*c*) Two lines of unlimited length, coinciding in part, coincide throughout.

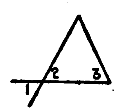
Ex. 60. Indicate the hypothesis and the conclusion of each of the following statements:

(*a*) If iron is heated, it expands.

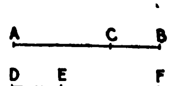
(*b*) If two angles of a triangle are equal, the opposite sides are equal.

(*c*) Two triangles are congruent if the sides of the one are respectively equal to the sides of the other.

(*d*) Vertical angles are equal.



Ex. 61. If in the annexed diagram $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$, for what reason* does $\angle 1$ equal $\angle 3$?



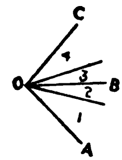
Ex. 62. If $AB = DF$, and $CB = DE$, for what reason does AC equal EF?

Ex. 63. In the following diagram, if $GH = IK$, for what reason does $GI = HK$?

Ex. 64. In the same diagram, if $GI = HK$, for what reason does GH equal IK?

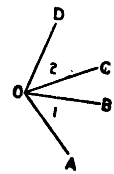
* The reasons requested in this and the following exercises are axioms.

Ex. 65. If OB bisects $\angle O$, and $\angle 1 = \angle 4$, why does $\angle 2$ equal $\angle 3$?



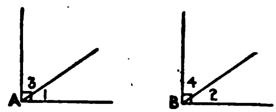
Ex. 66. In the same diagram, if $\angle 1 = \angle 4$, and $\angle 3 = \angle 2$, why does OB bisect $\angle O$?

Ex. 67. In the annexed diagram, if $\angle 1 = \angle 2$, why does $\angle AOC = \angle BOD$?

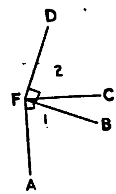


Ex. 68. In the same diagram, if $\angle AOC = \angle BOD$, why does $\angle 1 = \angle 2$?

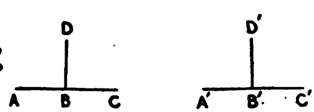
Ex. 69. In the annexed diagram, if $\angle A = \angle B$ and $\angle 1 = \angle 2$, why does $\angle 3$ equal $\angle 4$?



Ex. 70. In the annexed diagram, if $\angle AFC = 90°$, and $\angle BFD = 90°$, why does $\angle 1$ equal $\angle 2$?

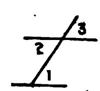


Ex. 71. If st. $\angle ABC =$ st. $\angle A'B'C'$, why does rt. $\angle DBC$ equal rt. $\angle D'B'C'$?



Ex. 72. If $\angle 1 + \angle b + \angle 2 = 180°$, $\angle 1 = \angle a$, and $\angle 2 = \angle c$, why does $\angle a + \angle b + \angle c = 180°$?

Ex. 73. If $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$, why does $\angle 1 = \angle 3$?



BOOK I

LINES AND RECTILINEAR FIGURES

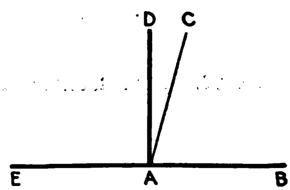
PRELIMINARY THEOREMS

46. *All right angles are equal.*

For all straight angles are equal (Ax. 15), and halves of equals are equal (Ax. 8).

47. *At a given point in a given line only one perpendicular can be drawn to the line.*

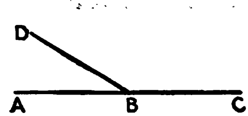
For if two perpendiculars, *AD* and *AC*, could be drawn at *A*, we should have two unequal right angles *BAD* and *BAC*, which is impossible (46).



48. *Complements of the same angle or of equal angles are equal.* (Ax. 3.)

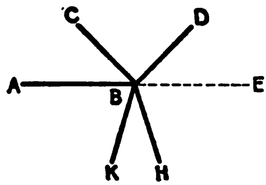
49. *Supplements of the same angle or of equal angles are equal.* (Ax. 3.)

50. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.* (Ax. 10.)



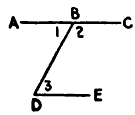
51. *If two adjacent angles are supplementary, their exterior sides are in the same straight line.*

Their exterior sides form a straight angle (36), and hence lie in a straight line (25).

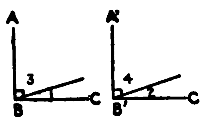


52. *The sum of all the angles about a point in a plane is equal to two straight angles.*

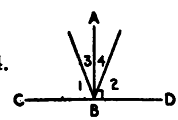
Ex. 74. In the annexed diagram, if ABC is a straight line, and $\angle 3$ is the supplement of $\angle 2$, why is $\angle 1 = \angle 3$?



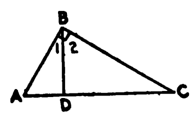
Ex. 75. If in the annexed diagram, $AB \perp BC$, $A'B' \perp B'C'$, and $\angle 3 = \angle 4$, prove $\angle 1 = \angle 2$.



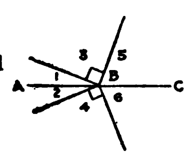
Ex. 76. If $AB \perp CD$ and $\angle 1 = \angle 2$, prove $\angle 3 = \angle 4$.



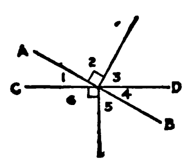
Ex. 77. If $\angle ABC$ is a right angle, and $\angle A$ is the complement of $\angle 1$, prove $\angle A = \angle 2$.

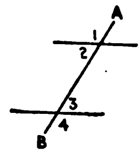


Ex. 78. If ABC is a st. $\angle$, $\angle 1 = \angle 2$, and $\measuredangle$ 3 and 4 are rt. $\measuredangle$, prove that $\angle 5 = \angle 6$.

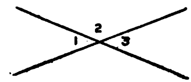


Ex. 79. If AB and CD are straight lines and $\measuredangle$ 2 and 6 are right angles, prove that $\angle 3 = \angle 5$.





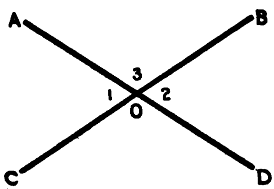
Ex. 80. In the annexed diagram, if AB is a straight line, and $\angle 2 = \angle 3$, prove that $\angle 1 = \angle 4$.



Ex. 81. If $\angle 1$ is the supplement of $\angle 2$, and $\angle 3$ is the supplement of $\angle 2$, why is $\angle 1 = \angle 3$?

PROPOSITION I. THEOREM

53. *Vertical angles are equal.*



Given the vertical ∠s 1 and 2.

To prove $\angle 1 = \angle 2$.

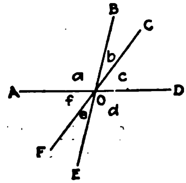
Proof

STATEMENTS	REASONS
$\angle 1$ is the sup. of $\angle 3$.	If two adj. ∠s have their ext. sides in a st. line, they are sup.
$\angle 2$ is the sup. of $\angle 3$.	For the same reason.
$\therefore \angle 1 = \angle 2$.	Supplements of the same ∠ are equal.
Q. E. D.	

54. Every proof consists of a number of statements, each of which is supported by a definite reason. The only admissible reasons are: a previously proved proposition; an axiom; a definition; or the hypothesis.

Ex. 82. In the diagram for the preceding proposition find $\angle 1$ (a) if $\angle 3 = 40°$, (b) if $\angle 3 = m°$.

Ex. 83. Three straight lines, AD, BE, and CF, meet in O, forming the six angles: a, b, c, d, e, and f.



(*a*) If $\angle b = 20°$, and $\angle c = 60°$, find $\angle AOE$.
(*b*) If $\angle FOB = 130°$, and $\angle c = 40°$, find $\angle a$.
(*c*) If $\angle FOD = 140°$, and $\angle b = 60°$, find $\angle a$.
(*d*) If $\angle f = 60°$, and $\angle b = 25°$, find $\angle d$.
(*e*) If $\angle$s b and f are complementary, find $\angle d$.
(*f*) If $\angle f = \angle b$, and $\angle d = 100°$, find $\angle b$.
(*g*) If $\angle a = 2(\angle c)$, and $\angle e = 60°$, find $\angle c$.
(*h*) If $\angle AOC = 140°$, and $\angle COE = 120°$, find $\angle BOD$.

Ex. 84. In the same diagram prove that

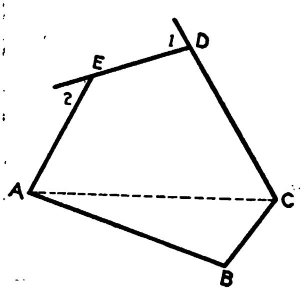
(*a*) $\angle b + \angle c = \angle AOE$.
(*b*) $\angle FOB - \angle c = \angle a$.
(*c*) $\angle FOB - \angle f = \angle d$.
(*d*) $\angle f + \angle b + \angle d = 180°$.
(*e*) $\angle AOC + \angle BOD + \angle COE = 360°$.
(*f*) $\angle AOC + \angle COE - \angle EOA = 2(\angle a)$.
(*g*) If $\angle f = \angle e$, then $\angle b = \angle c$.

* **Ex. 85.** In the same diagram

(*a*) If $\angle AOC = 150°$, and $\angle COE = 130°$, find $\angle a$.
(*b*) If $\angle FOB = 140°$, and $\angle AOC = 125°$, find $\angle d$.
(*c*) If $\angle AOE + \angle BOC = 140°$, and $\angle c = 40°$, find $\angle e$.
(*d*) If $\angle FOB = \angle FOD$, prove that $\angle f = \angle e$.
(*e*) Prove that reflex $\angle AOE$ + reflex $\angle BOF$ + reflex $\angle COA = 720°$.

55. Def. A **polygon** is a portion of a plane bounded by straight lines. The lines are called the **sides**. The **perimeter** of a polygon is the sum of all its sides.

The angles included by the adjacent sides are the **angles** of the polygon, and their vertices are the **vertices** of the polygon. An **exterior angle** is formed by a side and the prolongation of an adjacent one. A **diagonal** is a straight line joining the vertices of two non-adjacent angles.



* Exercises denoted by (*) are difficult and may be omitted in a first reading.

Thus $ABCDE$ is a polygon of five sides, AB and BC are sides, $\angle A$ is an angle, point B is a vertex, and AC is a diagonal of the polygon. $\angle 1$ and $\angle 2$ are exterior angles of the polygon.

56. Def. A **quadrilateral** is a polygon of four sides.

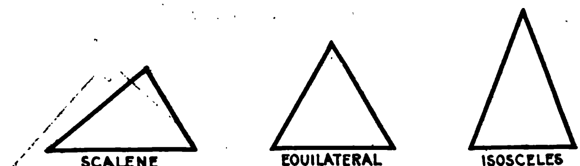
TRIANGLES — PART I

57. A **triangle** is a polygon of three sides.

58. **Triangles classified as to sides.** A **scalene** triangle is a triangle no two sides of which are equal.

An **isosceles** triangle is a triangle two sides of which are equal.

An **equilateral** triangle is a triangle all sides of which are equal.

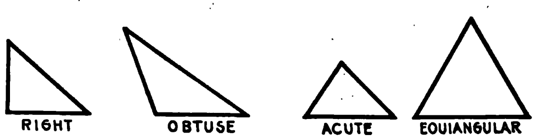


59. **Triangles classified as to angles.** A **right** triangle is a triangle one angle of which is *a right angle.*

An **obtuse** triangle is a triangle one angle of which is *an obtuse angle.*

An **acute** triangle is a triangle *all angles* of which are *acute.*

An **equiangular** triangle is a triangle *all angles* of which are equal.



60. Def. The **base** of a triangle is the side on which the figure appears to stand.

The two equal sides of an isosceles triangle are sometimes called the *arms*; the third side is called the *base*.

The angles opposite the arms are the *base angles* of an isosceles triangle.

61. Def. The **vertex angle** of a triangle is the angle opposite the base.

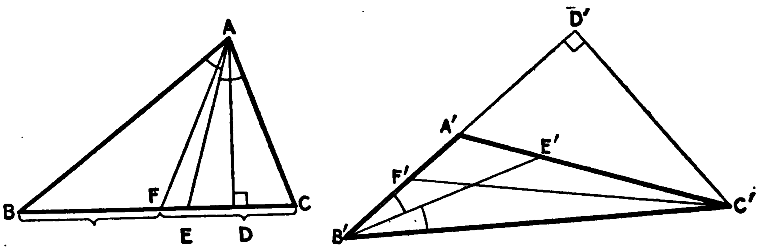
The vertex of this angle is sometimes called "the vertex of the triangle."

62. Def. The **hypotenuse** of a right triangle is the side opposite the right angle.

The sides including the right angle are sometimes called the *arms* of the right triangle.

63. Def. An **altitude** of a triangle is the perpendicular from any vertex to the opposite side (produced if necessary).

Thus, AD and $C'D'$ are altitudes. Every triangle has three altitudes.



64. Def. A **median** of a triangle is a line drawn from any vertex to the mid-point of the opposite side.

Thus, AF and $C'F'$ are medians. Every triangle has three medians.

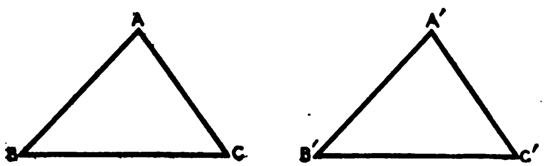
65. Def. An **angle-bisector** of a triangle is a line drawn from any vertex to the opposite side which bisects the corresponding angle.

Thus, AE and $B'E'$ are angle-bisectors. Every triangle has three angle-bisectors.

Note. Students who are familiar with the exercises on geometric drawing (39) should construct exactly a number of the diagrams defined above, as right $\triangle$, obtuse $\triangle$, medians, angle-bisectors, equilateral $\triangle$, etc.

PROPOSITION II. THEOREM

66. *Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.*



Given $\triangle$s ABC and $A'B'C'$ with $AB = A'B'$, $\angle A = \angle A'$, and $\angle B = \angle B'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

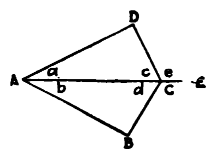
STATEMENTS	REASONS
Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB shall coincide with $A'B'$.	
Then BC will take the direction $B'C'$.	$\angle B = \angle B'$, by hyp.
AC will take the direction $A'C'$.	$\angle A = \angle A'$, by hyp.
$\therefore C$ will fall upon C'.	Two st. lines can intersect in only one point.
$\therefore \triangle ABC \cong \triangle A'B'C'$. Q. E. D.	Figures are congruent if they can be superposed.

67. This method of proof (**superposition**) is employed in fundamental propositions only. The student should place those parts upon each other whose equality is known, and, by successive steps, trace the position of the rest of the figure.

NOTE. In order to facilitate the citing of propositions, the following abbreviation is suggested for the above proposition: a. s. a. = a. s. a. Similar abbreviations will be suggested for other propositions.

68. Def. Polygons are **mutually equiangular** if their angles are respectively equal, and **mutually equilateral** if their sides are respectively equal.

If two polygons are mutually equiangular, lines or angles similarly situated are called **homologous lines or angles.** Thus AB and $A'B'$ (Prop. II) are homologous sides, C and C' homologous angles, the medians drawn from A and A' respectively homologous medians, etc.

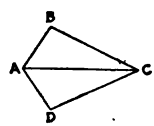


Ex. 86. **Hyp.** $\angle a = 30°$, $\angle b = 30°$, $\angle c = 60°$, $\angle d = 60°$.

To prove $\triangle ABC \cong \triangle ADC$.

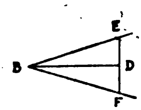
Ex. 87. **Hyp.** $\angle a = 40°$, $\angle b = 40°$, $\angle e = 130°$, $\angle d = 50°$, and ACE is a straight line.

To prove $\triangle ABC \cong \triangle ADC$.



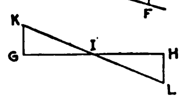
Ex. 88. **Hyp.** In quadrilateral $ABCD$, AC bisects angle A, and AC bisects $\angle C$.

To prove $\triangle ABC \cong \triangle AD'C$.



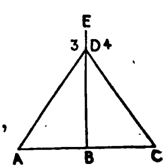
Ex. 89. **Hyp.** BD bisects $\angle B$,
$EF \perp BD$.

To prove $\triangle EBD \cong \triangle FBD$.



Ex. 90. **Hyp.** I is the mid-point of GH, $KG \perp GH$, $HL \perp GH$, KL is a straight line.

To prove $\triangle KGI \cong \triangle HLI$.



Ex. 91. **Hyp.** $DB \perp AC$, $\angle 3 = \angle 4$ and BDE is a straight line.

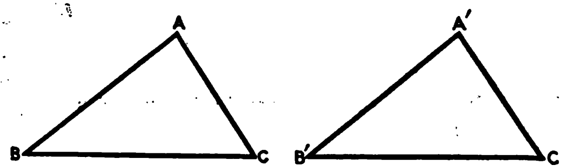
To prove $\triangle ABD \cong \triangle CBD$.

Ex. 92. **Hyp.** $DB \perp AC$, $\angle ADB$ is the sup. of $\angle 4$, and BDE is a st. line.

To prove $\triangle ABD \cong \triangle CBD$.

Proposition III. Theorem

69. *Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other* (*s.a.s.* = *s.a.s.*).



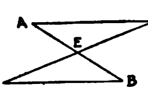
Given $\triangle$ ABC and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, and $\angle B = \angle B'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
Place $\triangle ABC$ upon $\triangle A'B'C'$ so that BC shall coincide with $B'C'$.	
BA will take the direction of $B'A'$,	$\angle B = \angle B'$, by hyp.
The point A will fall upon the point A',	$AB = A'B'$, by hyp.
$\therefore AC$ will coincide with $A'C'$.	One st. line and only one can be drawn through two points.
$\therefore \triangle ABC \cong \triangle A'B'C'$.	Figures are congruent if they can be made to coincide.
Q. E. D.	

Ex. 93. Hyp. Two lines AB and CD bisect each other in E.



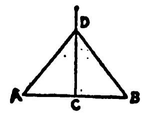
To prove $\triangle ADE \cong \triangle CEB$.*

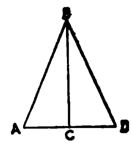
* The following set of exercises is of fundamental importance. No student should go on with the work until he is able to discover the proofs of all the simpler theorems given below. For additional work of this kind see Schultze's Teaching of Mathematics.

70. METHOD I. **The equality of lines and angles is usually proved by means of congruent triangles.**

Ex. 94. Hyp. DC is the perpendicular bisector of AB. (*I.e.* $AC = CB$ and $DC \perp AB$.)

To prove $AD = BD$.



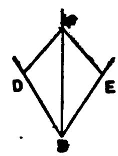


Ex. 95. **The bisector of the vertical angle of an isosceles triangle bisects the base.**

Ex. 96. Hyp. BF bisects $\angle DBE$.

$BD = BE$.

To prove $\angle FDB = \angle FEB$.

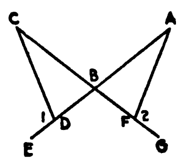


Ex. 97. Hyp. $BD = BF$, $BC = BA$, and CG and AE are st. lines.

To prove $CD = AF$.

Ex. 98. Hyp. $BD = BF$, $CD \perp EA$, $AF \perp CG$, and CG and AE are st. lines.

To prove $\angle C = \angle A$.



Ex. 99. Hyp. $BD = BF$, $\angle 1 = \angle 2$, and CG and AE are st. lines.

To prove $CB = AB$.

Ex. 100. Hyp. $CF = AD$, $DB = FB$, and CG and AE are st. lines.

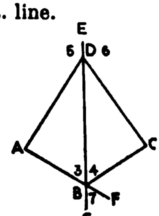
To prove $\angle C = \angle A$. (Preceding diagram.)

Ex. 101. Hyp. $AD = DC$, $\angle 5 = \angle 6$, and BE is a st. line.

To prove $\angle A = \angle C$.

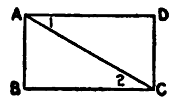
Ex. 102. Hyp. $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and BE is a st. line.

To prove $AD = DC$.



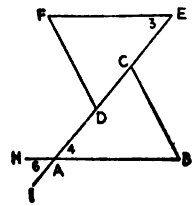
Ex. 103. Hyp. $AB = BC$, $\angle 4 = \angle 7$, and AF and EG are st. lines.

To prove $AD = DC$.



Ex. 104. Hyp. $AB \perp AD$, $DC \perp BC$ and $\angle 1 = \angle 2$.

To prove $\angle B = \angle D$.

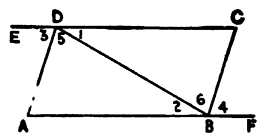


Ex. 105. Hyp. $AD = CE$, $AB = FE$, $\angle 3 = \angle 4$, and AE is a st. line.

To prove $\angle B = \angle F$.

Ex. 106. Hyp. $AD = CE$, $AB = FE$, $\angle 3 = \angle 6$, and EI and HB are st. lines.

To prove $BC = FD$.

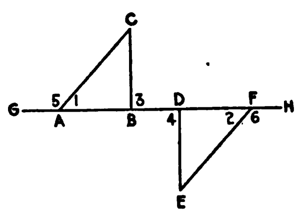


Ex. 107. Hyp. $\angle EDB = \angle FBD$, $\angle 5 = \angle 6$, and CE and AF are st. lines.

To prove $AD = BC$.

Ex. 108. Hyp. $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and EC and AF are st. lines.

To prove $AB = DC$.



Ex. 109. Hyp. $AD = BF$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and GH is a st. line.

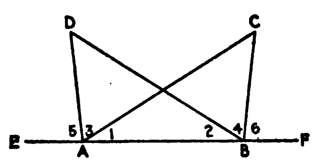
To prove $\angle C = \angle E$.

Ex. 110. Hyp. $AD = BF$, $\angle 5 = \angle 6$, $AC = FE$, and GH is a st. line.

To prove $BC = DE$.

Ex. 111. Hyp. $AD \perp EF$, $CB \perp EF$, $AD = BC$, and EF is a st. line.

To prove $\angle C = \angle D$.



Ex. 112. Hyp. $AD = CB$, $\angle 5 = \angle 6$, and EF is a st. line.

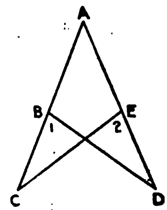
To prove $\angle 1 = \angle 2$.

Ex. 113. Hyp. $\angle 5 = \angle 6$, $\angle 1 = \angle 2$, and EF is a st. line.

To prove $AC = DB$.

Ex. 114. Hyp. $\angle 5 = \angle 6$, $\angle 3 = \angle 4$, and EF is a st. line.

To prove $AD = BC$.



Ex. 115. Hyp. $AB = AE$, $BC = ED$, and AC and AD are st. lines.

To prove $CE = BD$.

Ex. 116. Hyp. $AB = AE$, $\angle 1 = \angle 2$, and AC and AD are st. lines.

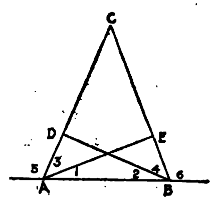
To prove $CE = BD$.

Ex. 117. Hyp. $AC = AD$, and $BC = ED$.*

To prove $\angle C = \angle D$.

Ex. 118. Hyp. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and AE and BD are st. lines.

To prove $AD = BE$.



Ex. 119. Hyp. $\angle DAB = \angle EBA$, and $\angle 3 = \angle 4$.

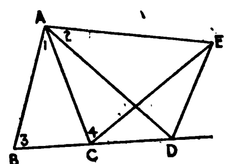
To prove $AD = BE$.

Ex. 120. Hyp. $\angle 5 = \angle 6$, $\angle 1 = \angle 2$, and AE and DB are st. lines.

To prove $AD = BE$.

Ex. 121. Hyp. $\angle 5 = \angle 6$, $AD = BE$, and AE and BD are st. lines.

To prove $\angle 1 = \angle 2$. (Preceding diagram.)



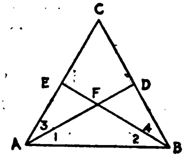
Ex. 122. Hyp. BCD is a st. line, $AB = AC$, $AD = AE$, and $\angle 1 = \angle 2$.

To prove $BD = CE$.

Ex. 123. Hyp. $AB = AC$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and AD is a st. line.

To prove $AD = AE$.

NOTE. The following exercises are more difficult because the two parts whose equality is to be demonstrated may be considered homologous parts of two different pairs of triangles, and the student has to determine by trial which pair must be used.



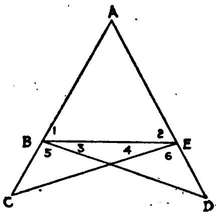
Ex. 124. Hyp. $\angle A = \angle B$, and $\angle 3 = \angle 4$.

To prove $AD = BE$.

Ex. 125. Hyp. $AC = BC$, and $\angle 3 = \angle 4$.

To prove $AD = BE$.

* If a line is denoted by two letters only, as AC or AD, it signifies that it is a straight line.



Ex. 126. **Hyp.** $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.
To prove $EC = BD$.

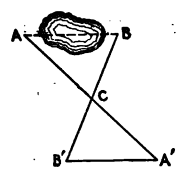
* **Ex. 127.** **Hyp.** $\angle 1 = \angle 2$, $DB \perp AC$, and $CE \perp AD$.
To prove $\angle C = \angle D$.

Ex. 128. **Hyp.** $AB = AE$, $\angle 5 = \angle 6$.
To prove $EC = BD$.

Ex. 129. If two angles of a triangle are equal, the corresponding angle-bisectors are equal.

Ex. 130. The medians drawn to the arms of an isosceles triangle are equal.

Ex. 131. If two exterior angles of a triangle are equal, the angle-bisectors of the adjacent interior angles are equal.

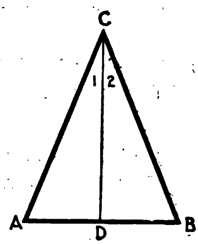


Ex. 132. Wishing to determine the distance across a pond, AB, we place a stake at a convenient station C, and by sighting from A, we locate A' in the prolongation of AC, and make $AC = CA'$. Similarly we produce BC to B' so that $BC = CB'$. Which line must we measure to obtain AB? Prove your statement.

[For additional practical applications see problems on page 285.]

Proposition IV. Theorem

71. *The bisector of the vertex-angle of an isosceles triangle divides the figure into two congruent triangles.*



Given $\triangle ABC$ with $AC = BC$, and $\angle 1 = \angle 2$.

To prove $\triangle ADC \cong \triangle DBC$.

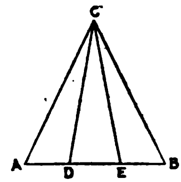
[The proof is left to the student.]

72. Cor. 1. **The base angles of an isosceles triangle are equal.**

73. Cor. 2. **An equilateral triangle is equiangular.**

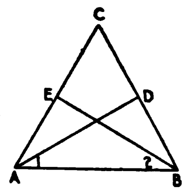
Ex. 133. In the annexed diagram, $AC = BC$, and $AE = DB$, then $CD = CE$.

Ex. 134. In the same diagram, if $AC = BC$, and $\angle DCA = \angle ECB$, then $CD = CE$.



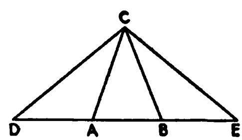
Ex. 135. In the annexed diagram, if $AC = BC$, and $\angle 1 = \angle 2$, then $AE = BD$.

Ex. 136. In the same diagram, if $AC = BC$, and AD and BE are angle-bisectors, then $AD = BE$.



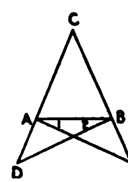
Ex. 137. In the annexed diagram, if $AC = BC$, $AD = BE$, and DE is a straight line, then DEC is an isosceles triangle.

Ex. 138. In the same diagram, if $AC = BC$, $\angle ACD = \angle BCE$, and DE is a straight line, then DEC is an isosceles triangle.



* **Ex. 139.** In the accompanying diagram, if $AC = BC$, $\angle 1 = \angle 2$, and CD and CE are straight lines, then $\angle D = \angle E$.

Ex. 140. If the base of an isosceles triangle is trisected, the lines joining the points of division with the vertex are equal.

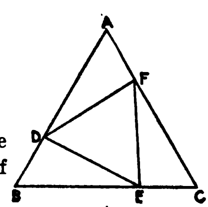


Ex. 141. If in the sides of an equilateral $\triangle ABC$, the points D, E, and F be taken so that

$$AD = BE = CF,$$

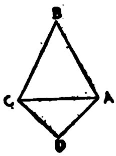
then $\triangle DFE$ is equilateral.

Ex. 142. Lines drawn from the mid-points of the arms of an isosceles triangle to the mid-point of the base are equal.

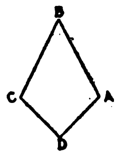


Ex. 143. In the preceding diagram, if $DE = EF = FD$, and $\angle AFD = \angle BDE = \angle CEF$, then $\triangle ABC$ is equilateral.

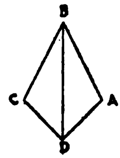
NOTE. The equality of angles is sometimes proved by means of Prop. IV (compare § 121).



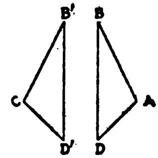
Ex. 144



Ex. 145



Ex. 146



Ex. 147

Ex. 144. If ABC and ADC are two isosceles triangles on the same base, AC, then $\angle BAD = \angle BCD$.

Ex. 145. If in quadrilateral $ABCD$, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.

Ex. 146. If two triangles ABD and DBC have BD common, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.

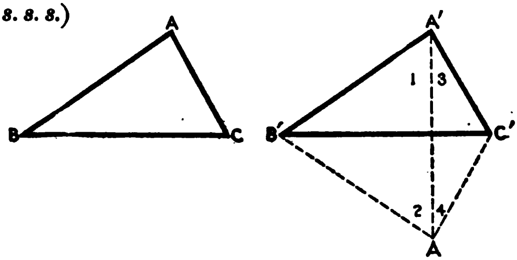
Ex. 147. If in two triangles ABD and $CB'D'$, $AD = CD'$, $AB = CB'$, and $BD = B'D'$, then $\angle A = \angle C$.

HINT. Place the $\triangle$ together, so as to form a quadrilateral.

PROPOSITION V. THEOREM

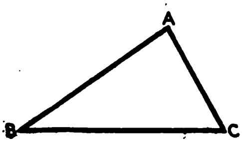
74. *Two triangles are congruent if three sides of one are respectively equal to three sides of the other.*

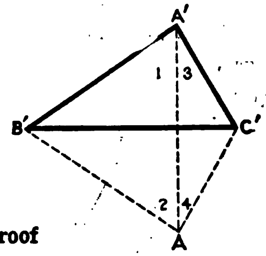
(s. s. s = s. s. s.)



Given $\triangle ABC$ and $A'B'C'$, with $AB = A'B'$, $BC = B'C'$, $AC = A'C'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.



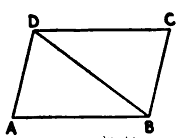


Proof

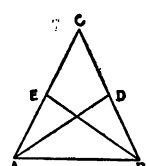
STATEMENTS	REASONS
Suppose no side longer than BC.	
Place $\triangle ABC$ so that BC shall coincide with $B'C'$ and A and A' lie on opposite sides of $B'C'$. Draw AA'.	
$\triangle AB'A'$ is isosceles.	$AB = A'B'$, by hyp.
$\therefore \angle 1 = \angle 2$.	The base $\angle$s of an isos. $\triangle$ are equal.
$\triangle AC'A'$ is isosceles.	$AC = A'C'$ by hyp.
$\therefore \angle 3 = \angle 4$.	The base $\angle$s of an isos. $\triangle$ are equal.
$\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$.	If equals are added to equals, the sums are equal.
Or $\angle A = \angle A'$.	Sub.
$\therefore \triangle AB'C' \cong \triangle A'B'C'$.	s. a. s = s. a. s.
I.e. $\triangle ABC \cong \triangle A'B'C'$.	
Q. E. D.	

Ex. 148. In the diagram opposite, if $AB = CD$ and $BC = DA$, then $\angle ABD = \angle BDC$.

Ex. 149. The median to the base of an isosceles triangle bisects the vertex-angle.



Ex. 148



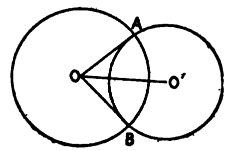
Ex. 150

Ex. 150. If $AE = BD$, $AD = BE$, and AC and BC are st. lines, then $\angle CEB = \angle CDA$.

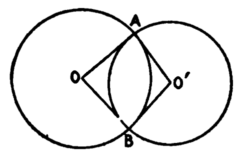
75. METHOD II. **If the lines or angles whose equality we wish to demonstrate are not parts of congruent triangles, we have to make them parts of congruent triangles by drawing additional lines.**

Ex. 151. If the opposite sides of a quadrilateral are equal, the opposite angles are equal.

Ex. 152. If two circles whose centers are O and O' intersect in A and B, then $\angle AOO' = \angle BOO'$.

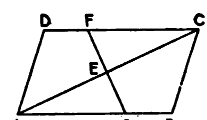


Ex. 153. If two circles whose centers are O and O' intersect in A and B, then $\angle OAO' = \angle OBO$.



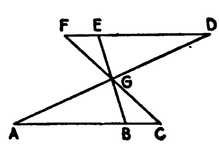
76. METHOD III. **If it is impossible to prove the congruence of the required pair of triangles, prove first the congruence of some other pair, or pairs, whose homologous parts will enable us to demonstrate the congruence of the original pair.**

Ex. 154. If the opposite sides of a quadrilateral are equal, and a line be drawn through the mid-point of the diagonal, terminating in two sides, this line is bisected.

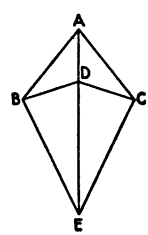


Ex. 155. In the annexed diagram, if $AG = GD$, $CG = FG$, and all lines are straight lines, then $BG = GE$.

Ex. 156. In the annexed diagram, if $AB = AC$, AE bisects $\angle BAC$, and AE is a st. line, then $\angle DBE = \angle DCE$.

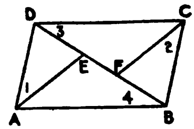


Ex. 155



Ex. 156

Ex. 157. In the annexed diagram, if $AB = CD$, $BC = DA$, and $\angle 1 = \angle 2$, and DB is a st. line, then $AE = CF$.



Ex. 158. In the same diagram, if $AB = CD$, $\angle 3 = \angle 4$ and $BF = DE$, then $\angle 1 = \angle 2$.

Ex. 159. Two triangles are equal if two sides and the median to one of these sides are equal respectively to two sides and the homologous median of the other.

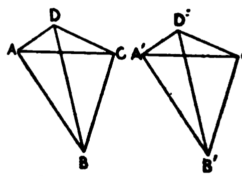
Ex. 160. Two isosceles triangles are equal if the vertex angle and the altitude upon one arm of one triangle are equal respectively to the vertex angle and the altitude upon one arm of the other.

Ex. 161. If the opposite sides of a quadrilateral are equal, the diagonals bisect each other.

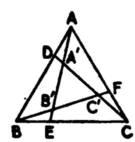
Ex. 162. Two triangles ABC and $A'B'C'$ are equal if $AB = A'B'$, $\angle A = \angle A'$, and angle-bisector AD = angle-bisector $A'D'$.

Ex. 163. If in quadrilateral $ABCD$, $AB = BC$, $CD = DA$, and diagonals AC and BD meet in E, then $AE = EC$.

Ex. 164. If in quadrilaterals $ABCD$, and $A'B'C'D'$, $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, and $AC = A'C'$, then $BD = B'D'$.

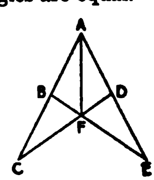


***Ex.** 165. If on the sides of an equilateral triangle, ABC, the points D, E, F are taken, so that $AD = BE = CF$, and E, D, and F are joined to the opposite vertices, then $\triangle A'B'C'$ is equilateral.



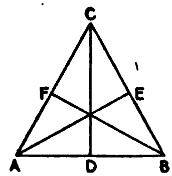
***Ex.** 166. If the opposite sides of a polygon of six sides are equal, and two of the opposite angles are equal, then all opposite angles are equal.

***Ex.** 167. In the annexed diagram, if $AB = AD$, $AC = AE$, and BE and DC are st. lines, then $\angle BAF = \angle DAF$.



* See footnote on page 20.

***Ex.** 168. In the annexed diagram, if $\angle A = \angle B$, and $AF = BE$, then $AD = DB$.



[For practical application see Ex. 3, page 285.]

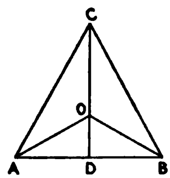
77. Method IV. **To prove that an angle is a right angle we usually demonstrate that it is equal to its supplementary adjacent angle.**

Ex. 169. The bisector of the vertex angle of an isosceles triangle is perpendicular to the base.

Ex. 170. The median to the base of an isosceles triangle is perpendicular to the base.

Ex. 171. If in quadrilateral $ABCD$ $AB = BC$, and $CD = DA$, then $BD \perp AC$.

Ex. 172. In the diagram opposite, if $AC = BC$, and $AO = BO$, then $CD \perp AB$.

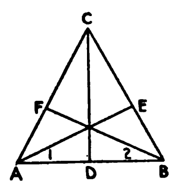


***Ex.** 173. In the diagram opposite, if

$$AC = BC,$$

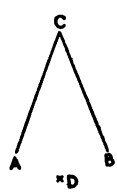
and $$\angle 1 = \angle 2,$$

then $$CD \perp AB.$$



---

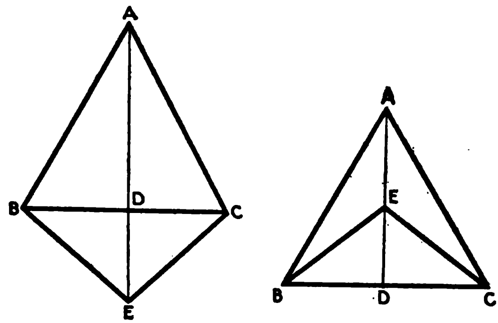
78. A point C is said to be **equidistant** from two other points A and B if $AC = BC$.



It is not necessary that these two lines be drawn. Thus, we may say D is equidistant from A and B, even if DA and DB are not drawn. In general, two letters, as AD, mean the straight line connecting A and D.

Proposition VI. Theorem

79. *The line that joins the vertices of two isosceles triangles on the same base bisects the common base at right angles.*



Given the isos. $\triangle$ ABC and BCE, with the common base BC.

To prove AE is the perpendicular-bisector of BC.

Proof

STATEMENTS	REASONS
In $\triangle$ ABE and CAE	
$AB = AC$,	Hyp.
$EB = EC$,	Hyp.
$AE = AE$.	Iden.
$\therefore \triangle BAE \cong \triangle CAE$.	s. s. s. = s. s. s.
In $\triangle$ ABD and ACD	
$AD = AD$,	Iden.
$AB = AC$,	Hyp.
$\angle BAD = \angle CAD$.	Hom. parts of congruent $\triangle$ are equal.
$\therefore \triangle ABD \cong \triangle ACD$.	s. a. s. = s. a. s.
$\therefore BD = DC$,	Hom. parts of congruent $\triangle$ are equal.

STATEMENTS	REASONS
and $\angle ADB = \angle ADC$.	Hom. parts of congruent $\triangle$ are equal.
I.e. $\angle ADB = \frac{1}{2}(\angle BDC)$.	
Or $\angle ADB = \frac{1}{2}(180°) = 90°$.	$\angle BDC$ is a st. $\angle$ by hyp.
Q. E. D.	

80. Cor. 1. Two points each equidistant from the ends of a line determine the perpendicular-bisector to that line.

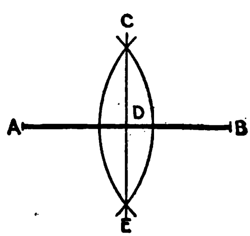
81. Cor. 2. Every point equidistant from the ends of a line lies in the perpendicular-bisector to that line.

Ex. 174. If the four sides of a quadrilateral are equal, the diagonals bisect each other.

CONSTRUCTIONS

Proposition VII. Problem

82. *To bisect a given straight line.*



Given a straight line AB.

Required to bisect AB.

Construction. From A and B as centers, with equal radii greater than $\frac{1}{2}$ AB, describe arcs intersecting at C and E.

Draw CE.

Then the line CE bisects AB at D.

Proof

STATEMENTS	REASONS
C is equidistant from A and B.	Con.
E is equidistant from A and B.	Con.
$\therefore AD = DB$. Q. E. D.	Two points each equidistant from the ends of a line determine the perpendicular-bisector to that line.

83. Cor. By means of the preceding construction we obtain also the perpendicular-bisector (CE) of a given line (AB).

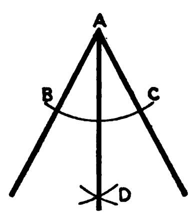
Ex. 175. Divide a given line into four equal parts.

Ex. 176. Construct the three medians of a triangle.

Ex. 177. Construct the three perpendicular-bisectors of the three sides of a given triangle.

Proposition VIII. Problem

84. *To bisect a given angle.*



Given $\angle CAB$.

Required to bisect $\angle CAB$.

Construction. From A as a center, with any radius, as AB, describe an arc cutting the sides of the $\angle A$ at B and C.

From B and C as centers, with equal radii greater than one half the distance from B to C, describe two arcs intersecting at D.

Draw *AD*.

AD is the required bisector.

Hint. What is the usual means of proving the equality of angles?

Note. It is advisable to draw the lines of construction either very thin or dotted, while the given and required lines are represented by heavy full lines. In complex constructions the resulting lines may be drawn in red or in blue.

Ex. 178. To divide a given angle into four equal parts.*

Ex. 179. To bisect a straight angle.

Ex. 180. Construct an angle of 90°, of 45°.

Ex. 181. Construct an angle of 22° 30′.

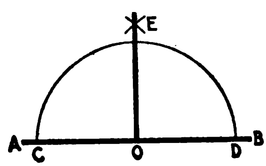
Ex. 182. Construct an angle of 67° 30′.

Ex. 183. Construct one half the supplement of a given angle.

Ex. 184. Construct the angle-bisectors of a given triangle. [For practical applications see page 286.]

Proposition IX. Problem

85. *At a given point in a given straight line, to erect a perpendicular to that line.*



Given point *O* in line *AB*.

Required a perpendicular to the line *AB* at *O*.

Construction. From *O* as a center, with any radius *OC*, describe an arc intersecting *AB* in *C* and *D*.

From *C* and *D* as centers, with equal radii greater than *OC*, describe two arcs intersecting at *E*.

* Some of the following exercises are identical with those given in § 39. Here, however, all problems should be proved.

Draw *OE*.

OE is the required perpendicular.

[The proof is left to the student.]

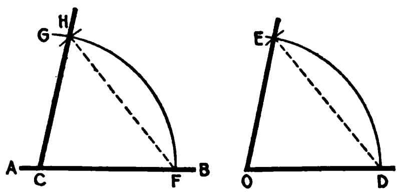
Ex. 185. Construct the complement of a given acute angle.

Ex. 186. Construct the supplement of the complement of a given acute angle.

Ex. 187. Construct one half the complement of a given acute angle.

PROPOSITION X. PROBLEM

86. *At a point in a given straight line to construct an angle equal to a given angle.*



Given point *C* in line *AB* and $\angle O$.

Required an $\angle$ at $C = \angle O$.

Construction. From *O* as a center, with any radius, describe an arc cutting the sides of $\angle O$ in *D* and *E*.

From *C* as a center, with the same radius, describe arc *FG*, intersecting *CB* in *F*. From *F* as a center, with a radius equal to *DE*, draw an arc intersecting arc *FG* in *H*.

Draw *CH*.

$\angle HCF$ is the required angle.

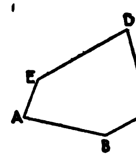
HINT. What is the usual means of proving the equality of angles?

NOTE. Two letters, *e.g. DE*, used as above, denote a straight line.

Ex. 188. Construct an angle equal to twice a given angle.

Ex. 189. Construct an angle equal to twice the supplement of a given angle.

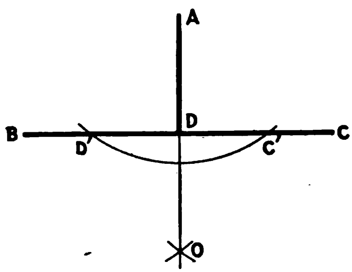
Ex. 190. Given a triangle ABC. Required another triangle $A'B'C'$, having $A'B' = AB$, $\angle A' = A$ and $\angle B' = B$. What is the relation of $\triangle ABC$ and $A'B'C'$?



Ex. 191. Given a polygon of five sides (pentagon) $ABCDE$. Required a pentagon having four of its sides equal respectively to AB, BC, CD, and DE, and the included angles respectively equal to $\angle s\ B$, C, and D.

PROPOSITION XI. PROBLEM

87. *From a point without a straight line, to let fall a perpendicular upon that line.*



Given a straight line BC, and a point A without the line.

Required a perpendicular from the point A to the line BC.

Construction. From A as a center, with a radius sufficiently great, describe an arc cutting BC in C' and D'.

From D' and C' as centers, with equal radii greater than $\frac{1}{2}D'C'$, describe two arcs intersecting at O.

Draw AO intersecting BC in D.

AD is the required perpendicular.

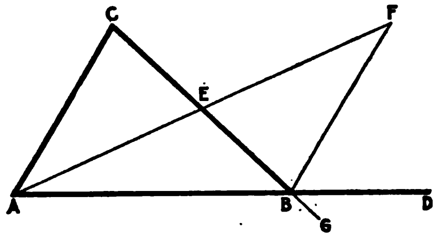
[The proof is left to the student.] (80)

Ex. 192. Construct the three altitudes of an acute triangle.

Ex. 193. Construct the three altitudes of an obtuse triangle.

PROPOSITION XII. THEOREM

88. *An exterior angle of a triangle is greater than either remote interior angle.*



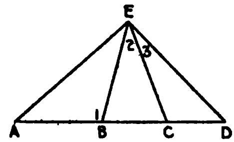
Given $\triangle ABC$ and the ext. $\angle CBD$.

To prove $\angle CBD > \angle A$ or $\angle C$.

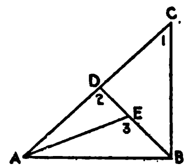
Proof

STATEMENTS	REASONS
Let E be the mid-point of BC. Draw AE and produce it its own length to F. Draw FB.	
In $\triangle$s ACE and FBE,	
$AE = EF$.	Con.
$EC = BE$.	Con.
$\angle CEA = \angle FEB$.	Vertical $\angle$s are equal.
$\therefore \triangle ACE \cong \triangle FBE$.	s. a. s. = s. a. s.
$\therefore \angle EBF = \angle C$.	Hom. parts of equal $\triangle$s.
But $\angle CBD > \angle EBF$.	The whole is greater than any of its parts.
Hence $\angle CBD > \angle C$.	Sub.
By joining the mid-point of AB to C, it follows in the same manner that	
$\angle ABG > \angle A$.	
But $\angle ABG = \angle CBD$.	Vertical $\angle$s are equal.
Hence $\angle CBD > \angle A$.	Sub.

Q. E. D.



Ex. 194. In the annexed diagram, if AD is a st. line, prove that (*a*) $\angle 1 > \angle 2$, (*b*) $\angle 1 > \angle D$, (*c*) $\angle 1 > \angle 3$.

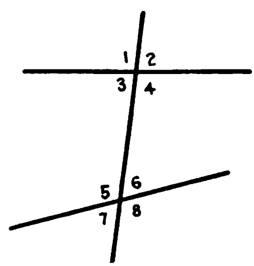


Ex. 195. In the annexed diagram if DB is a st. line, prove that (*a*) $\angle 2 > \angle 1$, (*b*) $\angle 3 > \angle 1$.

Ex. 196. In the diagram for Proposition XII, prove that

(*a*) $\angle FBD > \angle F$,
(*b*) $\angle BEA > \angle ACE$,
(*c*) $\angle CEA > \angle GBD$,
(*a*) $\angle CBD > \angle CEF$.

89. Def. A **transversal** is a line that intersects two (or more) other lines. The various angles formed by two lines and a transversal are named as follows:



1, 2, 7, 8 are **exterior** angles.

3, 4, 5, 6 are **interior** angles.

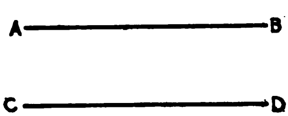
1 and 8, 2 and 7 are **alternate exterior** angles.

3 and 6, 4 and 5 are **alternate interior** angles.

1 and 5, 2 and 6, 3 and 7, 4 and 8 are **corresponding** angles.

PARALLEL LINES

90. Def. Two lines are **parallel** if they lie in the same plane and do not meet, however far produced (in either direction).

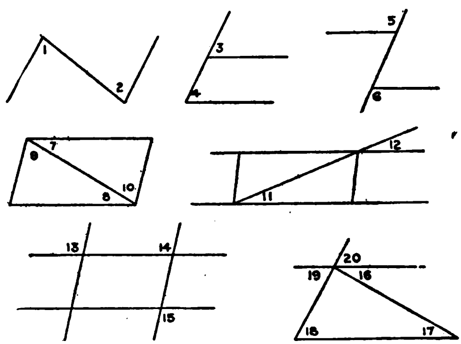


Thus, AB and CD represent two parallel lines.

91. Axiom 16. *Two intersecting lines cannot both be parallel to a third straight line.*

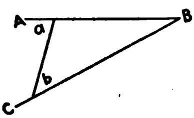
92. THEOREM. *Two straight lines which are parallel to a third straight line are parallel to each other.*

For if the two lines should meet, we should have two intersecting straight lines parallel to a third straight line, which contradicts Axiom 16.



Ex. 197. In the diagrams given here, tell what kind of angles the following are: 1 and 2, 3 and 4, 5 and 6, 7 and 8, 9 and 10, 11 and 12, 13 and 14, 14 and 15, 16 and 17, 18 and 19, 18 and 20.

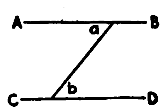
Ex. 198. In the diagram opposite, if AB and BC are st. lines, is it possible that



(*a*) $\angle a = 60°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 70°$?
(*c*) $\angle a = 60°$, and $\angle b = 60°$?

Ex. 199. In the next diagram, is it possible that the prolongations of AB and CD (*i.e.* toward the right) meet if

(*a*) $\angle a = 60°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 70°$?
(*c*) $\angle a = 60°$, and $\angle b = 60°$?



Ex. 200. In the same diagram, do the prolongations of BA and DC (*i.e.* toward the left) meet if

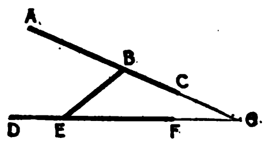
(*a*) $\angle a = 60°$, $\angle b = 70°$?
(*b*) $\angle a = 60°$, $\angle b = 50°$?
(*c*) $\angle a = 60°$, $\angle b = 60°$?

Ex. 201. In the same diagram, can the lines produced meet at all if

(*a*) $\angle a = 50°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 60°$?

PROPOSITION XIII. THEOREM

93. *Two lines are parallel if a transversal to these lines makes a pair of alternate interior angles equal.**



Given. AC and DF, intersected by BE so that $\angle ABE = \angle BEF$.

To prove $AC \parallel DF$.

Proof

STATEMENTS	REASONS
AC and DF either meet or are parallel.	Def. of $\parallel$ lines.
Suppose they meet in G.	
Then BEG is a triangle whose ext. $\angle ABE$ is equal to a remote int. $\angle BEG$, which is impossible.	An ext. $\angle$ of a $\triangle$ is greater than either remote int. $\angle$.
Hence AC and DF cannot meet.	
Or $AC \parallel DF$.	Def. of $\parallel$ lines.
Q. E. D.	

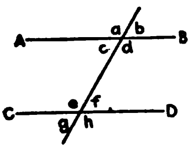
94. METHOD V. **To demonstrate that two lines are parallel, prove the equality of a pair of alternate interior angles.**

---

* In this proof we make a wrong assumption, viz., that two lines meet if the alt. int. $\angle$s are equal. The diagram which illustrates this wrong assumption must necessarily be inaccurate, and hence the two angles are not exactly equal. Similarly, all proofs which require the investigation of a wrong assumption (the so-called indirect proofs) are illustrated by inaccurate drawings, and it is illogical to attempt to make these drawings exact.

Ex. 202. Prove that AB and CD are parallel (see annexed diagram) if

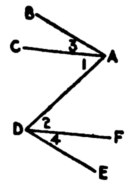
(a) $\angle c = 70^\circ$, $\angle f = 70^\circ$.
(b) $\angle c = 60^\circ$, $\angle e = 120^\circ$.
(c) $\angle a = 110^\circ$, $\angle f = 70^\circ$.
(d) $\angle b = 60^\circ$, $\angle f = 60^\circ$.
(e) $\angle a = 120^\circ$, $\angle g = 60^\circ$.



Ex. 203. Prove that $AB \parallel CD$, if $\angle b = \angle f$.

Ex. 204. Prove that $AB \parallel CD$, if $\angle a = \angle h$.

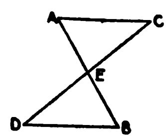
Ex. 205. Prove that $AB \parallel CD$, if $\angle b = \angle g$.



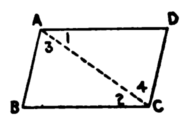
Ex. 206. In the diagram opposite, if $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, prove that $AB \parallel DE$.

Ex. 207. In the same diagram if $\angle A = \angle D$, and $\angle 3 = \angle 4$, prove that $AC \parallel DF$.

Ex. 208. In the same diagram if $BA \perp AD$, $ED \perp AD$, and $\angle 3 = \angle 4$, prove that $AC \parallel DF$.

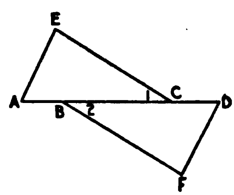


Ex. 209. If AB and CD bisect each other in E, prove that $AC \parallel DB$.



Ex. 210. In the diagram opposite, if $AB = DC$, and $\angle 3 = \angle 4$, prove that $AD \parallel BC$.

Ex. 211. In the same diagram, if $AB = DC$, and $AD = BC$, prove that $AB \parallel DC$.



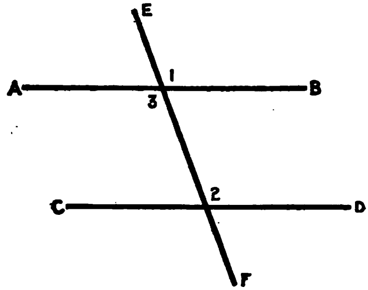
Ex. 212. In the diagram opposite, if $AB = CD$, $EC = BF$, $\angle 1 = \angle 2$, and AD is a st. line, then $AE \parallel DF$.

Ex. 213. In the same diagram, if $AE = DF$, $AB = CD$, and $EC = BF$, then $EC \parallel BF$.

Ex. 214. In the same diagram, if $AB = CD$, $\angle ABF = \angle DCE$, $EC = BF$, and AD is a straight line, then $AE \parallel DF$.

PROPOSITION XIV. THEOREM

95. *Two lines are parallel if a transversal to these lines makes a pair of corresponding angles equal.*



Given AB and CD, intersected by EF so that $\angle 1 = \angle 2$.

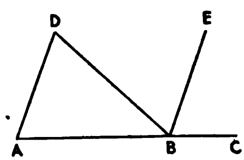
To prove $AB \parallel CD$

HINT. Prove the equality of a pair of alternate interior angles.

96. COR. If a transversal is perpendicular to two lines, these lines are parallel.

Ex. 215. In the diagram for Ex. 202, if $\angle a = 2(\angle b)$, and $\angle f = 60°$, then $AB \parallel CD$.

Ex. 216. In the annexed diagram, if $\angle A = 70°$, $\angle ABD = 60°$, $\angle DBE = 50°$, and AC is a straight line, prove that $AD \parallel BE$.

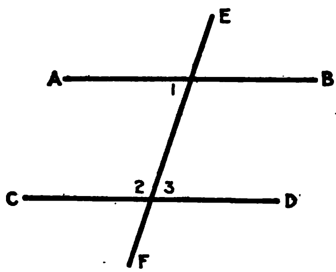


Ex. 217. In the same diagram, if BE bisects $\angle DBC$, $\angle ABD = 30°$, $\angle A = 75°$, and AC is a straight line, then $AD \parallel BE$.

Ex. 218. In the same diagram, if $AB = BD$ and $\angle D = \angle EBC$, then $AD \parallel BE$.

PROPOSITION XV. THEOREM

97. *Two lines are parallel if a transversal to these lines makes a pair of interior angles on the same side of the transversal supplementary.*



Given AB and CD, intersected by EF so that

$$\angle 1 + \angle 2 = 180°.$$

To prove $AB \parallel CD$.

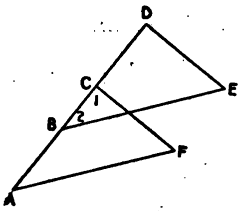
Proof

STATEMENTS	REASONS
$\angle 1$ is the sup. of $\angle 2$.	Hyp.
$\angle 3$ is the sup. of $\angle 2$.	If two adj. ∠s have their ext. sides in a st. line, they are sup.
$\therefore \angle 1 = \angle 3$.	Supplements of the same ∠ are equal.
$\therefore AB \parallel CD$. Q. E. D.	Lines are ∥ if a pair of alt. int. ∠s are equal.

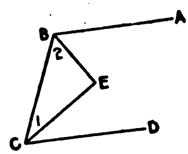
98. METHOD VI. **Sometimes lines are demonstrated to be parallel by proving that**

(*a*) **Two corresponding angles are equal, or**

(*b*) **Two interior angles on the same side of a transversal are supplementary.**



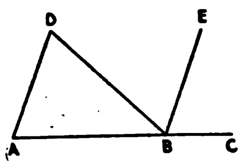
Ex. 219.



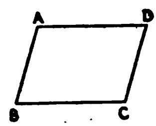
Ex. 220.

Ex. 219. In the diagram above, if $AF = BE$, $AB = CD$, $\angle A = \angle 2$, and AD is a st. line, prove that $CF \parallel DE$.

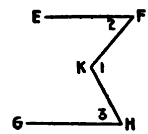
Ex. 220. In the diagram above, if BE bisects $\angle B$, CE bisects $\angle C$, and $\angle 1 + \angle 2 = 90°$, then $BA \parallel CD$.



Ex. 221.



Ex. 222.



Ex. 223.

Ex. 221. In the diagram opposite, if $AB = BD$ and $\angle D$ is the supplement of $\angle ABE$, then $AD \parallel BE$.

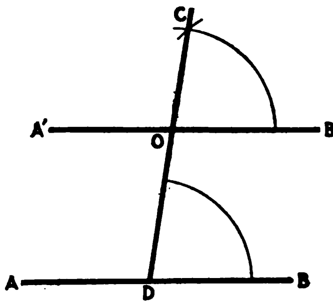
Ex. 222. If $\angle A + \angle B + \angle C + \angle D = 360°$, $\angle A = \angle C$, and $\angle B = \angle D$, then $AD \parallel BC$.

* **Ex.** 223. If $\angle 1 = \angle 2 + \angle 3$, then $EF \parallel GH$.

99. To **construct figures** that bear certain relations we employ the same methods as for proving these relations. Thus, to draw a line parallel to another, we may construct a pair of equal alt. int. $\angle$s; to make an angle equal to another angle we use a construction (86) which really involves the construction of congruent triangles, etc.

Proposition XVI. Problem*

100. *Through a given point to draw a parallel to a given line.*



Given line AB and point O.

Required a line through $O \parallel AB$.

Construction. Through O draw DC, intersecting AB in D.
At O construct $\angle COB' = \angle ODB$. (86)
$A'B'$ is the required line.

Hint. What are the means of proving that two lines are parallel?

Note. If previous constructions are employed in a problem, the details of such construction should *not* be stated. Thus, in the preceding problem it would be awkward and illogical to describe the method of making $\angle COB' = \angle ODB$.

The beginner, however, should *draw* all such details. In other words the construction should be carried out exactly by means of ruler and compasses.

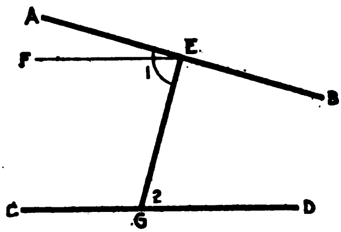
Ex. 224. Through a given point to draw a line parallel to a given line, by means of equal alt. int. angles.

Ex. 225. Through a given point to draw a line parallel to a given line by means of Ex. 209 (p. 46).

* See practical application, page 286.

Proposition XVII. Theorem

101. *Two lines are not parallel if a transversal to these lines makes a pair of alternate interior angles unequal.*



Given AB and CD cut by a transversal, so that

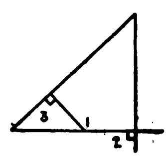
$$\angle 1 > \angle 2.$$

To prove AB is not $\parallel$ CD.

Proof

STATEMENTS	REASONS
Construct $\angle FEG = \angle 2$.	
$EF \parallel CD$	Two lines are $\parallel$ if a transversal makes the alt. int. $\angle$s equal.
$\therefore AB$ is not $\parallel$ CD. Q. E. D.	Two intersecting lines cannot both be $\parallel$ to a third line.

102. Cor. Two lines respectively perpendicular to two intersecting lines are not parallel.



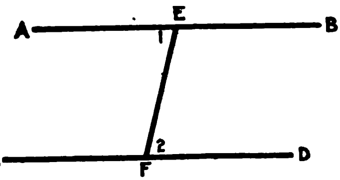
Hint. Prove $\angle 1 > \angle 2$.

103. A proposition is the **converse** of another when the hypothesis and the conclusion of the one are respectively the conclusion and the hypothesis of the other.

Proposition XVIII. Theorem

104. *If two parallels are cut by a transversal, the alternate interior angles are equal.*

[Converse of Prop. XIII.]



Given parallel lines AB and CD, and the alt. int. $\angle$s 1 and 2.

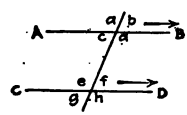
To prove $\angle 1 = 2$.

Proof

STATEMENTS	REASONS
Either $\angle 1 = \angle 2$, or $\angle 1 \neq \angle 2$.	
Suppose that $\angle 1 \neq \angle 2$, then AB is not $\parallel$ to CD.	Two lines are not $\parallel$ if a transversal to these lines makes a pair of alt. int. $\angle$s unequal.
But this is impossible.	$AB \parallel CD$, by hyp.
Hence $\angle 1 = \angle 2$. Q. E. D.	

105. Cor. If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other also.

Ex. 226. Two $\parallel$ lines AB and CD are cut by a transversal.

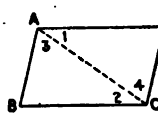


(a) If $\angle c = 70°$, find $\angle f$.
(b) If $\angle c = 80°$, find $\angle e$.
(c) If $\angle c = 75°$, find $\angle g$.

Ex. 227. In the same diagram, if $AB \parallel CD$, and
(*a*) $\angle a = 110°$, find $\angle f$.
(*b*) $\angle b = 80°$, find $\angle g$.
(*c*) $\angle h = 100°$, find $\angle b$.

Ex. 228. In the same diagram, if $AB \parallel CD$, prove that
(*a*) $\angle a = \angle e$.
(*b*) $\angle b = \angle f$.
(*c*) $\angle a = \angle h$.
(*d*) $\angle c$ is the sup. of $\angle e$.
(*e*) $\angle g$ is the sup. of $\angle d$.

Ex. 229. In the diagram opposite if $AD = BC$, and $AD \parallel BC$, then $\angle 3 = \angle 4$.

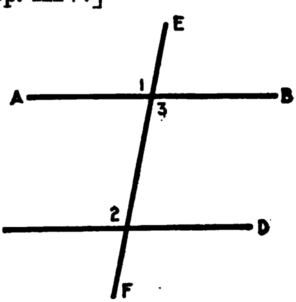


Ex. 230. In the same diagram, if $AB \parallel CD$, and $AD \parallel BC$, then $\angle B = \angle D$.

PROPOSITION XIX. THEOREM

106. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

[Converse of Prop. XIV.]

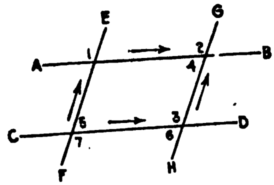


Given parallel lines AB and CD and the cor. $\angle$s 1 and 2.

To prove $\angle 1 = \angle 2$.

Proof

STATEMENTS	REASONS
$\angle 1 = \angle 3$.	Vertical $\angle$s are equal.
$\angle 2 = \angle 3$.	Alt. int. $\angle$s of $\parallel$ lines are equal.
$\therefore \angle 1 = \angle 2$. Q.E.D.	Things equal to the same thing are equal to each other.

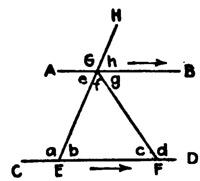


Ex. 231. If $AB \parallel CD$ and $EF \parallel GH$, prove that

(*a*) $\angle 1 = \angle 3$.
(*b*) $\angle 4 = \angle 5$.
(*c*) $\angle 2 = \angle 7$.

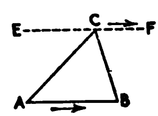
Ex. 232. If $AB \parallel CD$, and EH is a st. line, find the numerical value of

(*a*) $\angle d$, if $\angle g = 80°$.
(*b*) $\angle a$, if $\angle h = 60°$.
(*c*) $\angle FGH$, if $\angle b = 60°$, and $\angle c = 70°$.
(*d*) $\angle b$, if $\angle FGH = 140°$, and $\angle c = 80°$.
(*e*) $\angle f$, if $\angle b = 65°$, and $\angle c = 70°$.
(*f*) $\angle b$, if $\angle f = 50°$, and $\angle c = 70°$.
(*g*) $\angle HGF$, if $\angle e = 55°$, and $\angle c = 75°$.

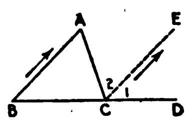


Ex. 233. In the same diagram, if $AB \parallel CD$, prove that

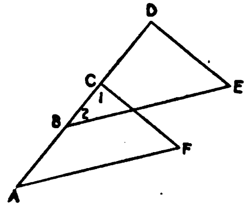
(*a*) $\angle FGH = \angle b + \angle c$.
(*b*) $\angle f = 180° - \angle h - \angle c$.
(*c*) $\angle CEH = \angle f + \angle c$.
(*d*) $\angle b + \angle f + \angle c = 180°$.



Ex. 234. Prove that the sum of the three angles of $\triangle ABC$ equals 180°.



Ex. 235. If ACD is an ext. angle of $\triangle ABC$, then $\angle ACD = \angle A + \angle B$.



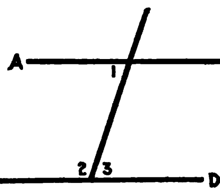
Ex. 236. In the annexed diagram, if AD is a st. line, $AB = CD$, $AF \parallel BE$, and $CF \parallel DE$, then $DE = CF$.

Ex. 237. In the same diagram, if $AB = CD$, $CF = DE$, $CF \parallel DE$, and AD is a st. line, then $AF = BE$.

PROPOSITION XX. THEOREM

107. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

[Converse of Prop. XV.]

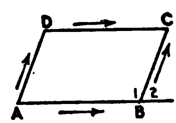


Given $AB \parallel CD$, and the int. $\angle$s 1 and 2, lying on the same side of a transversal.

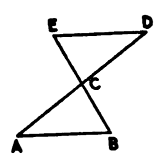
To prove $\angle 1$ is the sup. of $\angle 2$.

Proof

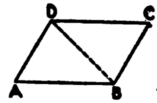
STATEMENTS	REASONS
$\angle 3$ is the sup. of $\angle 2$.	If two adj. $\angle$s have their ext. sides in a st. line, they are sup.
$\angle 1 = \angle 3$.	Alt. int. $\angle$s of $\parallel$ lines.
$\therefore \angle 1$ is the sup. of $\angle 2$. Q. E. D.	Sub.



Ex. 238. If $AB \parallel CD$, and $AD \parallel BC$, prove
(*a*) $\angle 2$ is the sup. of $\angle D$.
(*b*) $\angle A = \angle C$.

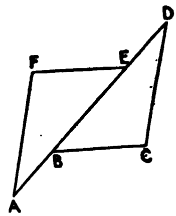


Ex. 239. In the annexed diagram, if $ED \parallel AB$, $AC = CD$, and EB and AD are st. lines, $\triangle ABC \cong \triangle CDE$.



Ex. 240. If the opposite sides of a quadrilateral are parallel, they are also equal.

Ex. 241. If two sides of a quadrilateral are equal and parallel, the other two sides are equal.

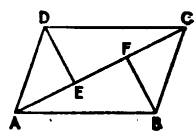


Ex. 242. If in the diagram given here $AB = ED$, $BC \parallel FE$, $AF \parallel DC$, and AD is a st. line, $\triangle AEF$ must equal $\triangle BCD$.

Ex. 243. In the same diagram, if $AB = ED$, $BC = FE$, $BC \parallel FE$, and AD is a st. line, AF must equal CD.

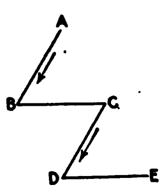
Ex. 244. In the same diagram, if $AB = ED$, $AF = DC$, $AF \parallel DC$, and AD is a st. line, then BC is parallel to FE.

Ex. 245. If two sides of a quadrilateral are equal and parallel, the other two sides are parallel.

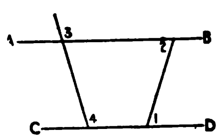


Ex. 246. In the diagram given here, if $AB \parallel CD$, $AD \parallel BC$, $AE = FC$, and AC in a st. line, then $DE \parallel BF$.

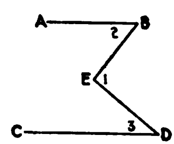
Ex. 247. The bisectors of a pair of corresponding angles formed by parallel lines are parallel.



Ex. 248



Ex. 249

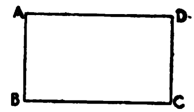


Ex. 250

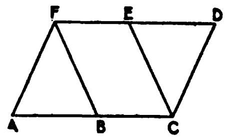
Ex. 248. If in the diagram given here, $AB \parallel CD$, and $\angle B = \angle D$, prove that $BC \parallel DE$.

Ex. 249. In the diagram given here, if $\angle 1 = \angle 2$, then $\angle 3 = \angle 4$.

* **Ex. 250.** If in the diagram above, $AB \parallel CD$, prove that $\angle 1 = \angle 2 + \angle 3$.



* **Ex. 251.** If in the diagram opposite, ∡ A, B, and C are right angles, $\angle D$ is also a right angle.

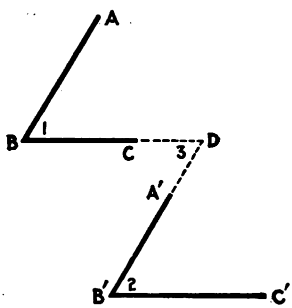


* **Ex. 252.** If in the diagram opposite, $AC \parallel FD$, $AF \parallel CD$, and $FB \parallel EC$, prove that

$$\triangle AFB \cong \triangle ECD.$$

PROPOSITION XXI. THEOREM.

108. *Two angles whose corresponding sides are parallel and extend in the same direction from their vertices are equal.*

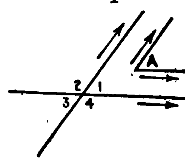


Given $AB \parallel AB'$ and $BC \parallel B'C'$.

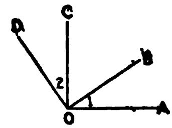
To prove $\angle B = \angle B'$.

HINT. Produce if necessary BC and $B'A'$ until they meet at D.

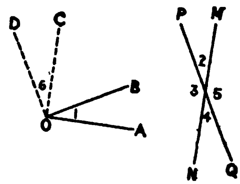
109. COR. Angles whose corresponding sides are parallel are either equal or supplementary.



Thus, in the diagram opposite, if the corresponding sides are parallel, ∡ 1 and 3 equal $\angle A$, and ∡ 2 and 4 are sup. of $\angle A$.



Ex. 253. If $CO \perp AO$, and $DO \perp OB$, then $\angle 1 = \angle 2$.



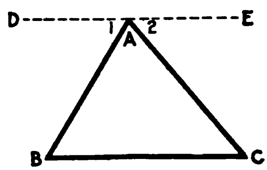
Ex. 254. Two angles are either equal or supplementary if the sides of one are respectively perpendicular to the sides of the other (as 1 and 2, or 1 and 3, or 1 and 4, or 1 and 5).

HINT. Draw $\angle 6$ whose sides are parallel to MN and PQ respectively.

TRIANGLES — PART II

PROPOSITION XXII. THEOREM

110. *The sum of the angles of a triangle is equal to a straight angle.*



Given $\triangle ABC$.

To prove $\angle A + \angle B + \angle C =$ a st. $\angle$.

Proof

STATEMENTS	REASONS
Through A draw $DE \parallel BC$.	
$\angle 1 = \angle B$.	Alt. int. $\angle$s of $\parallel$ lines.
$\angle 2 = \angle C$.	Alt. int. $\angle$s of $\parallel$ lines.
But	
$\angle A + \angle 1 + \angle 2 =$ a st. $\angle$.	DAC is a st. line.
$\therefore \angle A + \angle B + \angle C =$ a st. $\angle$.	Sub.
Q. E. D.	

111. Cor. 1. In a triangle there can be only one obtuse angle or one right angle.

112. Cor. 2. The acute angles of a right triangle are complementary.

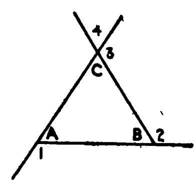
113. Cor. 3. If two triangles have two angles of the one respectively equal to two angles of the other, the third angles are equal.

114. Cor. 4. Two triangles are congruent if two angles and the side opposite one of them are equal respectively to two angles and the homologous side of the other (s. a. a. = s. a. a.).

115. Cor. 5. From a point without a line there can be only one perpendicular to that line.

116. Cor. 6. Each angle of an equiangular triangle is equal to sixty degrees.

Ex. 255. In the annexed diagram find the value of



(a) $\angle A$, if $\angle B = 60°$, and $\angle C = 50°$.
(b) $\angle B$, if $\angle A = m°$, and $\angle C = n°$.
(c) $\angle 3$, if $\angle A = 50°$, and $\angle B = 70°$.
(d) $\angle 4$, if $\angle 2 = 110°$, and $\angle A = 60°$.
(e) $\angle 1$, if $\angle 4 = 40°$, and $\angle B = 70°$.
(f) $\angle 3$, if $\angle 2 = 140°$, and $\angle 1 = 120°$.
(g) $\angle A + \angle B$ if $\angle C = m°$.

Ex. 256. If C is the vertex angle of isosceles triangle ABC, find

(a) $\angle A$, if $\angle C = 40°$.
(b) $\angle B$, if $\angle C = m°$.
(c) $\angle C$, if $\angle A = 40°$.
(d) $\angle C$, if $\angle B = n°$.

Ex. 257. How many degrees are there in each angle of an isosceles right triangle?

Ex. 258. If two angles of a triangle are 60° and 40° respectively, what is the angle formed by the bisectors of these angles?

Ex. 259. If two angles of a triangle are given, construct the third.

Ex. 260. If a base angle of an isosceles triangle is given, construct the vertex angle.

Ex. 261. If the vertex angle of an isosceles triangle is given, construct a base angle.

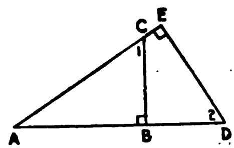
Ex. 262. Construct an angle of 60°.

Ex. 263. Construct an angle of 30°.

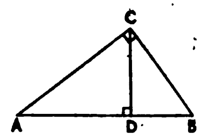
Ex. 264. Construct an angle of 120°; of 75°.

Ex. 265. The bisectors of two interior angles on the same side of a transversal to two parallel lines are perpendicular to each other.

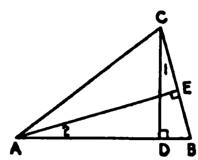
Ex. 266. If CB and ED are drawn respectively perpendicular upon the sides of angle A, prove $\angle 1 = \angle 2$.



Ex. 267. If CD is the altitude upon the hypotenuse of right triangle ABC, prove that $\angle ACD = \angle B$.



Ex. 268. If CD and AE are altitudes of triangle ABC, prove that $\angle 1 = \angle 2$.



Ex. 269. Find the sum of the four angles of a quadrilateral.

Ex. 270. If two angles of a triangle are equal, the bisector of the third angle divides the figure into two equal triangles.

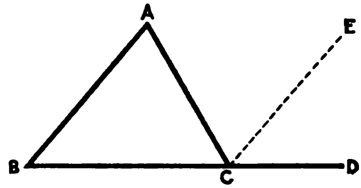
Ex. 271. Two right triangles are equal if the hypotenuse and an acute angle of the one are equal respectively to the hypotenuse and an acute angle of the other.

Ex. 272. The altitudes upon the arms of an isosceles triangle are equal.

Ex. 273. The lines drawn from the mid-point of the base of an isosceles triangle perpendicular to the arms are equal.

PROPOSITION XXIII. THEOREM

117. *An exterior angle of a triangle is equal to the sum of the two remote interior angles.*



Given the ext. $\angle ACD$ of $\triangle ABC$.

To prove $\angle ACD = \angle A + \angle B$.

HINT. Draw $CE \parallel BA$.

118. COR. An exterior angle of a triangle diminished by a remote interior angle equals the other remote interior angle.

Ex. 274. Given two angles of a triangle, construct an angle equal to the remote exterior angle.

Ex. 275. Given an exterior angle and a remote interior angle of a triangle. Construct an angle equal to the other remote interior angle.

Ex. 276. If two angles of a triangle are equal, the bisector of the remote exterior angle is parallel to the opposite side of the triangle.

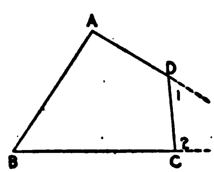
Ex. 277. If at each vertex of a triangle, one exterior angle is drawn, the sum of the three angles is equal to four right angles.

Ex. 278. The sum of two exterior angles of a triangle diminished by the third interior angle is equal to two right angles.

* **Ex. 279.** If the sum of two exterior angles of a triangle is equal to three right angles, the triangle is a right one.

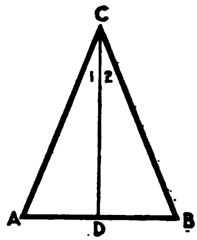
* **Ex. 280.** The bisectors of two exterior angles of a triangle include an angle equal to half the third exterior angle.

* **Ex. 281.** The sum of two interior angles of a quadrilateral is equal to the sum of the two nonadjacent exterior angles. (*I.e.* $\angle A + \angle B = \angle 1 + \angle 2$.)



Proposition XXIV. Theorem

119. *If two angles of a triangle are equal, the sides opposite these angles are equal.*



Given $\triangle ABC$, with $\angle A = \angle B$.

To prove $AC = BC$.

Hint. Construct CD, the bisector of angle ACB, and prove the congruence of the triangles.

120. Cor. An equiangular triangle is also equilateral.

121. Method VII. **Propositions IV and XXIV are occasionally used to prove the equality of lines or the equality of angles.** (Compare note on p. 31.)

Ex. 282. If two exterior angles, formed by producing one side of a triangle at both ends, are equal, the other sides are equal.

Ex. 283. If the vertex angle of an isosceles triangle is 60°, the triangle is equilateral.

Ex. 284. If the bisector of an exterior angle of a triangle is parallel to one side, the triangle is isosceles.

Ex. 285. The bisectors of the base angles of an isosceles triangle form, if they meet, an isosceles triangle.

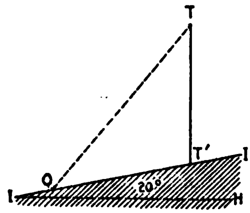
Ex. 286. If at the ends of the base of an isosceles triangle perpendiculars are erected upon the arms, another isosceles triangle is formed.

Ex. 287. A line parallel to the base of an isosceles triangle, intersecting the two arms, cuts off another isosceles triangle.

Ex. 288. If in quadrilateral $ABCD$, $AB = BC$, and $\angle A = \angle C$, then $CD = DA$.

Ex. 289. If OA and OB are two radii of a circle, and a point C within the circle is equidistant from A and B, then $\angle OAC = \angle OBC$.

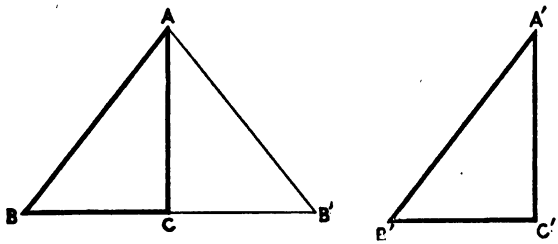
Ex. 290. A tower TT' stands on a declivity II' which is inclined 20° to a horizontal line IH (*i.e.* $\angle I = 20°$). At a point O, 60 feet from T', $\angle TOT' = 35°$. Find TT'.



[For additional practical applications see problems 7–18, p. 286.]

PROPOSITION XXV. THEOREM

122. *Two right triangles are congruent if the hypotenuse and an arm of the one are respectively equal to the hypotenuse and an arm of the other.* (*hy. arm = hy. arm.*)



Given $\triangle$s ABC and $A'B'C'$, with $AB = A'B'$; $AC = A'C'$, and the rt. $\angle$s C and C'.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
Place $\triangle A'B'C'$ so that $A'C'$ coincides with AC, and that B and B' fall on opposite sides of AC.	
BCB' is a st. line.	$\angle BCB = 2$ rt. $\angle$s or a st. $\angle$.
$AB = A'B'$.	Hyp.
$\therefore \angle B = \angle B'$.	Base $\angle$s of isos. $\triangle$ are equal.
$\therefore \triangle ABC \cong \triangle A'B'C'$.	s. a. a = s. a. a.
Q. E. D.	

Ex. 291. If the perpendiculars from the mid-point of one side of a triangle upon the two other sides are equal, the triangle is isosceles.

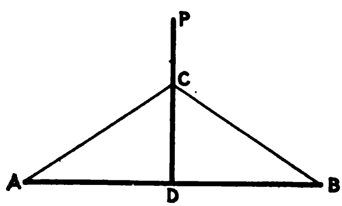
Ex. 292. If two altitudes of a triangle are equal, the corresponding sides are equal, and the triangle is isosceles.

Ex. 293. Two triangles are congruent if two sides and the altitude upon one of them of one triangle are respectively equal to two sides and the homologous altitude of the other.

Ex. 294. Two triangles are congruent if the base, the median, and altitude to the base of one triangle are equal respectively to the base, the median, and altitude to the base of the other.

Proposition XXVI. Theorem

123. *Every point in the perpendicular-bisector of a line is equidistant from the extremities of that line.*



Given PD, the perpendicular-bisector of AB, and C any point in PD.

To prove C is equidistant from A and B.

Hint. What is the usual means of proving the equality of lines?

Ex. 295. In a given line AB find a point equidistant from two given points P and Q.

Ex. 296. In a given circumference find a point equidistant from two given points.

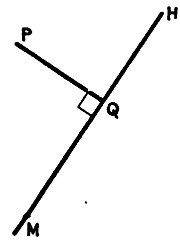
Ex. 297. Find a point equidistant from three given points.

Ex. 298. Given three fixed points A, B, and C. Required to construct a point X, so that $AX = BX$, and $CX = \frac{1}{2}$ inch.*

* For applied problems see problems 19 and 20, p. 288.

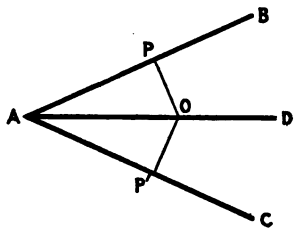
124. Def. The **distance** of a point from a line is the length of the perpendicular from the point to the line.

Thus, if PQ is perpendicular to MN, PQ is called the distance of P from MN.



Proposition XXVII. Theorem

125. *Every point in the bisector of an angle is equidistant from the sides of the angle.*



Given AD, the bisector of $\angle BAC$, and O, any point in AD.

To prove O is equidistant from AB and AC.

Hint. Draw $OP \perp AB$ and $OP' \perp AC$, and prove the congruence of the two triangles.

126. Cor. A point equidistant from the sides of an angle is in the bisector of that angle.

Ex. 299. In a given line AB to find a point equidistant from the sides of the given $\angle DEF$.

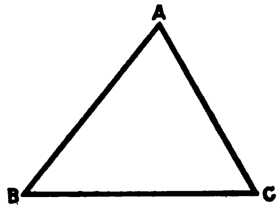
Ex. 300. In a given circumference find a point equidistant from two given lines.

Ex. 301. Find a point equidistant from three given lines.

UNEQUAL LINES AND UNEQUAL ANGLES

PROPOSITION XXVIII. THEOREM

127. *The sum of two sides of a triangle is greater than the third side.*

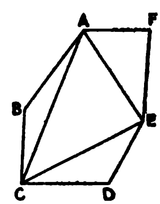


Given $\triangle ACB$.

To prove $AB + AC > BC$.

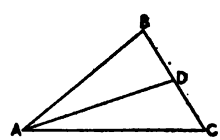
HINT. See Axioms.

Ex. 302. Is it possible to draw a triangle whose sides are
(*a*) 4 in., 5 in., 10 in.? (*b*) 8 in., 19 in., 12 in.?
(*c*) 4 ft., 12 ft., 8 ft.? (*d*) 6 in., 2 in., 5 in.?



Ex. 303. Prove that the perimeter of polygon $ABCDEF >$ perimeter of $\triangle ACE$.

Ex. 304. If in $\triangle ABC$, $AC = BC$, and a point D in the prolongation of AC be connected with B, then $DA > DB$.



Ex. 305. If in the side BC of $\triangle ABC$ any point D be taken, then

$$AB + BC > AD + DC.$$

Ex. 306. If from any point E in $\triangle ABC$ EB and EC are drawn, then

$$AB + AC > EB + EC.$$

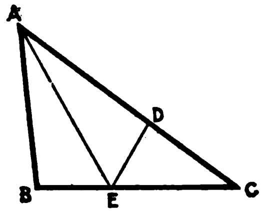
HINT. Produce BE until it meets AC in D.

Ex. 307. The difference of any two sides of a triangle is less than the third side.

Ex. 308. A diagonal of a quadrilateral is less than one half the perimeter of the figure.

PROPOSITION XXIX. THEOREM

128. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



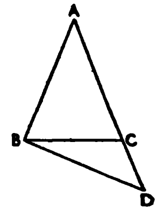
Given $\triangle ABC$, with $AC > AB$.

To prove $\angle B > \angle C$.

Proof

STATEMENTS	REASONS
Draw the angle-bisector AE.	
On AC lay off $AD = AB$ and draw DE.	
In $\triangle$s ABE and ADE	
$AB = AD$.	Con.
$AE = AE$.	Iden.
$\angle BAE = \angle DAE$.	Con.
$\therefore \triangle ABE \cong \triangle ADE$.	s. a. s. = s. a. s.
$\therefore \angle B = \angle ADE$.	Homologous parts of congruent $\triangle$s.
But $\angle EDA > \angle C$.	Ext. $\angle$ of a $\triangle >$ either remote int. $\angle$.
$\therefore \angle B > \angle C$.	Sub.
Q. E. D.	

Ex. 309. If in $\triangle ABC$, $AB = 7$, $BC = 8$, $CA = 6$, which is the greatest angle of the triangle? which the smallest?



Ex. 310. If one arm AC of an isosceles triangle ABC is produced to D, then

$$\angle ABD > \angle ADB.$$

Ex. 311. If in $\triangle ABC$, $AB > AC$, and $\angle B = 60°$, prove that $\angle C > \angle A$.

Ex. 312. If in $\triangle ABC$, $AB > AC$, and $\angle A = 60°$, which is the greatest angle of the triangle?

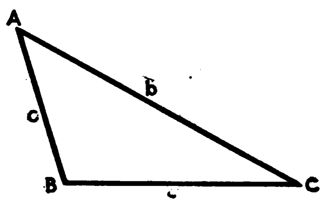
Ex. 313. If in quadrilateral $ABCD$, $AB > BC$ and $AD > CD$, then $\angle C > \angle A$.

129. NOTE. The sides of a triangle are often designated by italics which correspond with the letters of the opposite vertices. Thus, in $\triangle ABC$, $AB = c$, $BC = a$, and $CA = b$.

PROPOSITION XXX. THEOREM

130. *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*

[Converse of Prop. XXIX.]



Given $\triangle ABC$, with $\angle B > \angle C$.

To prove $b > c$.

Proof

STATEMENTS	REASONS
$b < c$, or $b = c$, or $b > c$.	
Suppose $b < c$,	
then $\angle B < \angle C$.	In a $\triangle$, the greater $\angle$ lies opposite the greater side.

STATEMENTS	REASONS
But this is impossible.	$\angle B > \angle C$, by hyp.
Suppose $b = c$, then $\angle B = \angle C$.	The base ∡ of an isos. $\triangle$ are equal.
But this is impossible.	$\angle B > \angle C$, by hyp.
$\therefore b > c$.	
Q. E. D.	

131. Cor. The perpendicular is the shortest line that can be drawn from a point to a given line. (115.)

Note. The method used in the above proof is known as the **indirect method** or **reductio ad absurdum.** Instead of showing that a certain conclusion is true, we examine *all* conclusions which contradict the one to be proved, and demonstrate that these are false. Thus, in order to demonstrate $a = b$, we simply prove that the statement $a \neq b$ is false. Or to prove $m > n$, we disprove the only contradictory conclusions, viz. $m = n$, and $m < n$.

The indirect proof is often used to prove converses, especially those that establish inequalities. (Compare Props. XII and XVII.)

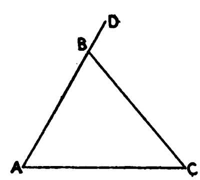
Ex. 314. Which is the greatest side of a right triangle? of an obtuse triangle?

Ex. 315. If two angles of a triangle are 50° and 60° respectively, which is the greater of the two opposite sides? which is the greatest side of the triangle? which is the shortest?

Ex. 316. If in $\triangle ABC$, $AB > AC$ and $\angle B = 60°$, which is the greatest side of the triangle? which is the smallest?

Ex. 317. If in $\triangle ABC$, $AB > AC$, and $\angle A = 60°$, which is the greatest side of the triangle?

Ex. 318. Of two lines drawn from a point in a perpendicular to a given line, cutting off on the given line unequal segments from the foot of the perpendicular, the more remote is the greater.

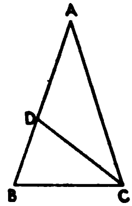


Ex. 319. If $\angle A$ of $\triangle ABC$ equals two thirds of a right angle, and the exterior angle DBC equals 110°, which is the greatest side of the triangle? which is the shortest?

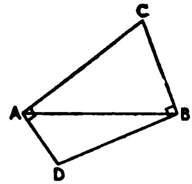
Ex. 320. If in $\triangle ABC$ $AB=AC$, prove that $DC>DB$.

Ex. 321. In triangle ABC, $AC>BC$. If the bisectors of angles A and B meet in D, prove $AD>BD$.

Ex. 322. In the diagram given here, $AC>BC$, $AD\perp AC$, and $DB\perp BC$. Prove $BD>AD$.



Ex. 320



Ex. 322

Ex. 323. Prove Prop. XXX directly.

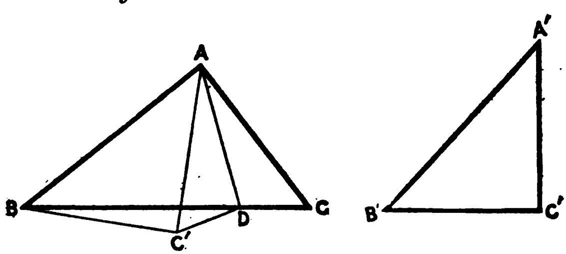
HINT. At B draw angle $CBD=\angle C$.

Ex. 324. The sum of the altitudes of a triangle is less than the perimeter of the triangle.

[See practical problems, p. 289, No. 21.]

PROPOSITION XXXI. THEOREM

132. *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given $\triangle$s ABC and $A'B'C'$ with

$$AB=A'B';\ AC=AC'';\ \angle A>\angle A'.$$

To prove $BC>B'C'$.

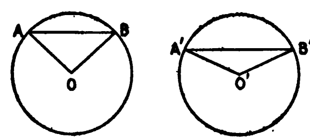
Proof

STATEMENTS	REASONS
Apply $\triangle A'B'C'$ to $\triangle ABC$ so that $A'B'$ coincides with AB.	$AB = A'B'$, by hyp.
$A'C'$ will fall between AB and AC as AC'.	$\angle A > \angle A'$, by hyp.
Draw AD bisecting $\angle CAC'$ to meet BC in D, and draw DC'.	
$AC' = AC$,	Hyp.
$\angle C'AD = \angle CAD$,	Con.
$AD = AD$.	Iden.
$\therefore \triangle C'AD \cong \triangle CAD$.	s. a. s. = s. a. s.
$\therefore DC' = DC$.	Hom. parts of congruent $\triangle$s.
But $BD + DC' > BC'$,	The sum of 2 sides of a $\triangle >$ third side.
$\therefore BD + DC > BC'$,	Sub.
or $BC > BC'$ or $B'C'$.	Sub.
Q. E. D.	

Ex. 325. If O and O' are two equal circles, and

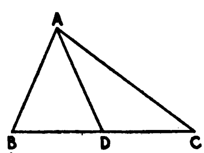
$$\angle A'O'B' > \angle AOB,$$

then $$A'B' > AB.$$



Ex. 326. If in triangle ACB the median AD be drawn forming the acute angle ADB, prove

(a) $AC > AB$
(b) $\angle B > C$.

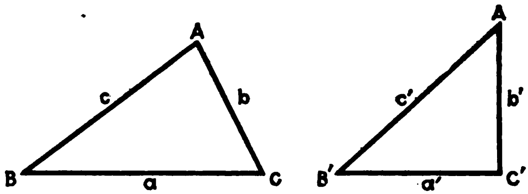


Ex. 327. If in $\triangle ABC$, $\angle A = \angle B$, and a point D in AB be taken so that $\angle ACD > DCB$, then $AD > DB$.

Proposition XXXII. Theorem

133. *If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second,* *then the included angle of the first is greater than the included angle of the second.*

[Converse of Prop. XXXI.]



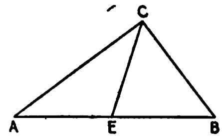
Given $\triangle$ ABC and $A'B'C'$, with $b = b'$; $c = c'$; $a > a$.

To prove $\angle A > \angle A'$.

Proof

STATEMENTS	REASONS
$A < A'$, or $A = A'$, or $A > A'$.	
First, suppose $A < A'$.	
Then $a < a'$.	If two $\triangle$ have 2 sides of the one equal respectively to 2 sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, then the 3d side of the first $>$ 3d side of the second.
But this is impossible.	$a > a'$, by hyp.
Second, suppose $A = A'$.	
Then $\triangle ABC \cong \triangle A'B'C'$.	*s. a. s.* = *s. a. s.*
$\therefore a = a'$.	Homologous parts, etc.
But this is impossible.	$a > a'$, by hyp.
$\therefore A > A'$.	
Q. E. D.	

134. NOTE. In the demonstrations which follow, the reasons will, as a rule, not be given in detail, but merely by reference to the paragraph number. The student, however, should in many cases state the proofs completely, with reasons in full, although occasionally the mere "statements" will be sufficient to test the pupil's understanding.



Ex. 328. If in $\triangle ABC$, having $AC > BC$, the median CE is drawn, then $\angle AEC$ is an obtuse angle.

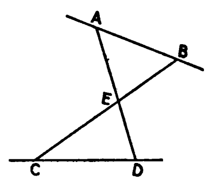
Ex. 329. Given $\triangle ABC$, having $\angle A = \angle B$. If the point D in AB be taken so that $AD > DB$, then $\angle ACD > \angle DCB$.

135. METHOD VIII. **Inequality of lines is generally proved by means of Props. XXVIII, XXIX, XXXI. If it is impossible to discover any relation of the angles, Prop. XXVIII is used; if the two lines are parts of the same triangle, and the opposite angles can be proved unequal, Prop. XXIX is used; but if the sides or angles are parts of different triangles, Prop. XXXI is used. In some cases several methods will lead to the desired result.**

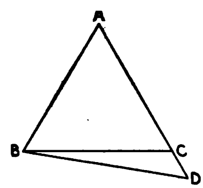
The inequality of angles is proved in a similar manner by Props. XII, XXX, and Prop. XXXII.

Ex. 330. If AD and BC intersect in E, prove $AD + BC < AB + CD$. (Which of the three propositions must be used?)

Ex. 331. If AC, a side of an equilateral triangle, is produced to D, and BD is joined, then $AD > BD > AB$.



Ex. 330

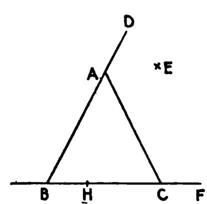


Ex. 331

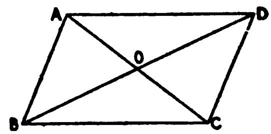
Ex. 332. A point P, not in the perpendicular-bisector of AB, is unequally distant from A and B. (Find 3 proofs corresponding with the methods of 135.)

Ex. 333. In the annexed diagram, if $AB = AC$, prove that

(*a*) $BD > DC$.
(*b*) $BE > EC$.
(*c*) $AF > AB$.
(*d*) $AB > AH$.



Ex. 334. The sum of the diagonals of any quadrilateral is less than the perimeter, but greater than the semiperimeter of the quadrilateral.



*** Ex. 335.** If the opposite sides of a quadrilateral $ABCD$ are equal, but $AB < AD$, prove that $\angle AOB < \angle AOD$.

*** Ex. 336.** The sum of the lines drawn from any point in a triangle to its vertices is less than the perimeter, but greater than the semiperimeter of the triangle. (Ex. 306.)

*** Ex. 337.** If in $\triangle ABC$, having $AB < BC$, BD, the bisector of angle B, is drawn, meeting AC in D, then $AD < DC$.

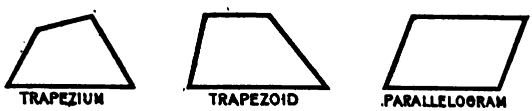
*** Ex. 338.** If in $\triangle ABC$, having $AC > BC$, the median CD is drawn, then any point E in BD is nearer to B than to A.

QUADRILATERALS

136. Def. A **trapezium** is a quadrilateral having no two sides parallel.

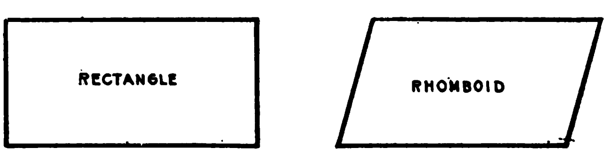
A **trapezoid** is a quadrilateral having two, and only two, sides parallel.

A **parallelogram** is a quadrilateral having its opposite sides parallel.



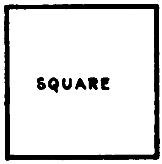
137. Def. A **rectangle** is a parallelogram whose angles are right angles.

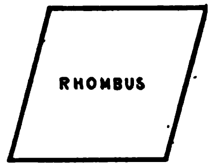
A **rhomboid** is a parallelogram whose angles are oblique.



138. Def. A **square** is an equilateral rectangle.

A **rhombus** is an equilateral rhomboid.





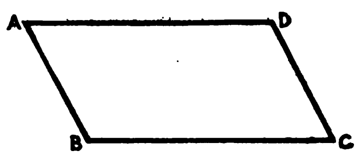
139. Def. An **isosceles trapezoid** is a trapezoid whose non-parallel sides are equal. The parallel sides of a trapezoid are called its **bases**, and are distinguished as **upper** and **lower**.

The **median** of a trapezoid is the line joining the mid-points of the non-parallel sides.

140. Def. A **diagonal** of a quadrilateral is a straight line joining opposite vertices. The **altitude** of a parallelogram or trapezoid is the perpendicular distance between the two bases.

Proposition XXXIII. Theorem

141. *The opposite sides and angles of a parallelogram are equal.*



Given $\square ABCD$.

To prove $AD = BC$; $AB = CD$; $\angle A = \angle C$; $\angle B = \angle D$.

Hint. What is the usual means of proving the equality of lines and angles?

142. Cor. 1. A diagonal divides a parallelogram into two congruent triangles.

143. Cor. 2. If one angle of a parallelogram is a right angle, the figure is a rectangle.

144. Cor. 3. Parallels included between parallels are equal.

145. Cor. 4. If two adjacent sides of a parallelogram are equal, the figure is equilateral, and hence either a rhombus or a square.

Ex. 339. The perpendiculars to a diagonal of a parallelogram from the opposite vertices are equal.

Ex. 340. Let $ABCD$ be a parallelogram and BE and DF lines drawn perpendicular to AD and BC. Prove $BE = DF$.

Ex. 341. If in parallelogram $ABCD$ the diagonals meet in E, then $AE = EC$.

Ex. 342. Let $ABCD$ be a parallelogram and BD a diagonal, prove that the angle-bisectors AE and CF of triangles ABD and CBD respectively are equal.

Ex. 343. If the points G and E trisect the diagonal BD of parallelogram $ABCD$, prove that $AG = CE$.

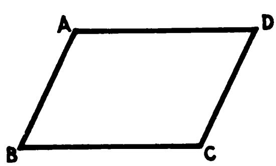
Ex. 344. The diagonals of a rectangle are equal.

Ex. 345. The diagonals of a rhomboid are unequal.

Ex. 346. If the diagonals of a parallelogram are equal, the figure is a rectangle.

Proposition XXXIV. Theorem

146. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given quadrilateral $ABCD$, with

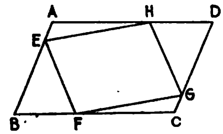
$$AB = CD; \ AD = BC.$$

To prove $AD \parallel BC; \ AB \parallel CD.$

Hint. Prove the equality of a pair of alternate interior angles by means of congruent triangles.

Ex. 347. If the four sides of a quadrilateral are equal, the figure is either a rhombus or a square.

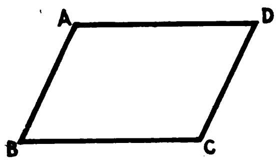
Ex. 348. If *ABCD* is a parallelogram, $AE = CG$, and $BF = DH$, then *EFGH* is a parallelogram.



Ex. 349. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Proposition XXXV. Theorem

147. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given quadrilateral *ABCD*, with

$AB \parallel CD$; and $AB = CD$.

To prove *ABCD* is a parallelogram.

[The proof is similar to that in Prop. XXXIV.]

Ex. 350. If on opposite sides of *AB* the parallelograms *ABCD* and *ABEF* are drawn, then *CDEF* is also a parallelogram.

Ex. 351. If two opposite sides of a parallelogram are produced by the same length in opposite directions and their ends joined to the nearest vertices, another parallelogram is formed.

148. Method IX. **Lines may be shown to be parallel by proving them to be opposite sides of a parallelogram.**

Ex. 352. If two opposite sides of a parallelogram are divided into three equal parts, and the respective points of division are joined, the lines are parallel.

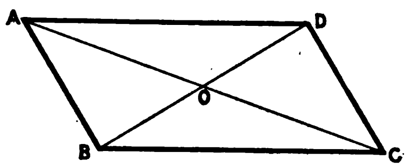
Ex. 353. If two opposite sides of a parallelogram are produced by the same length in the same direction, a line joining the ends is parallel to the other sides of the parallelogram.

Ex. 354. If from two opposite vertices of a parallelogram lines be drawn bisecting two opposite sides, respectively, the lines are parallel.

[For practical problems, see p. 289.]

Proposition XXXVI. Theorem

149. *The diagonals of a parallelogram bisect each other.*



Given ▱ $ABCD$.

To prove $AO = OC$, $BO = OD$.

[The proof is left to the student.]

Ex. 355. In the diagram of Prop. XXXVI let E be the mid-point of BO, and F the mid-point of OD. Prove that $AECF$ is a parallelogram.

Ex. 356. The diagonals of a rhombus are perpendicular to each other.

Ex. 357. If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus, or a square.

Ex. 358. If each half of the diagonals of a parallelogram is bisected and the mid-points are joined in order, another parallelogram is formed.

Ex. 359. If a diagonal bisects an angle of a parallelogram, the figure is equilateral.

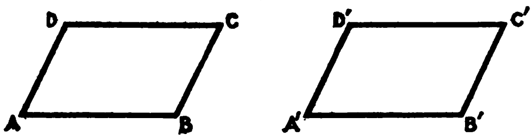
Ex. 360. If the ends of two diameters of a circle be joined in succession, a rectangle is formed.

Ex. 361. The base angles of an isosceles trapezoid are equal.

Ex. 362. State and prove the converse of the preceding exercise.

Proposition XXXVII. Theorem

150. *Two parallelograms are congruent if two adjacent sides and the included angle of one are equal, respectively, to two adjacent sides and the included angle of the other.*



Given ▱s $ABCD$ and $A'B'C'D'$, with $AB = A'B'$, $AD = A'D'$ $\angle A = \angle A'$

To prove ▱ $ABCD =$ ▱ $A'B'C'D'$.

Proof

STATEMENTS	REASONS
Place ▱ $ABCD$ upon $A'B'C'D'$, so that AB coincides with $A'B'$.	
Then AD takes the direction $A'D'$.	$\angle A = \angle A'$, by hyp.
And D coincides with D'.	$AD = A'D'$, by hyp.
BC takes the direction of $B'C'$.	Axiom 16.
Or C must lie in $B'C'$ or in $B'C'$ produced.	
Similarly, C must lie in $D'C'$ or in $D'C'$ produced.	Axiom 16.
Hence C coincides with C', and ▱ $ABCD$ coincides with ▱ $A'B'C'D'$.	Axiom 13 *b*.
I.e. ▱ $ABCD \cong$ ▱ $A'B'C'D'$. Q. E. D.	

Ex. 363. If two opposite sides of a parallelogram are trisected and the corresponding points are joined, the figure is divided into three equal parallelograms.

151. Method X. **To prove that a line is twice as large as another we usually double the smaller, and prove that its double equals the greater, or sometimes we bisect the greater, and prove that its half equals the smaller. The same relation between angles is proved in a similar way.**

Ex. 364. If two angles of a triangle are 30° and 60° respectively, the side opposite the angle of 30° is one half the side included by the two angles.

Ex. 365. In triangle ABC, if through the mid-point D of AC a line DE is drawn parallel to AB, and E lies in CB, then DE is one half AB.

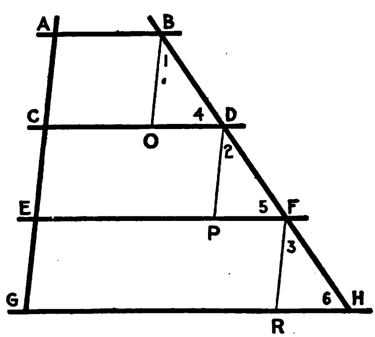
Ex. 366. The median to the hypotenuse of a right triangle is one half the hypotenuse.

Ex. 367. The angle formed by the base of an isosceles triangle and the altitude upon one arm is one half the vertical angle.

* **Ex. 368.** If in quadrilateral $ABCD$, $AB \parallel DC$, $CD = DA$, and $\angle D = 2\,(\angle B)$ then CD is one half AB.

Proposition XXXVIII. Theorem *

152. *If three or more parallels intercept equal lengths on one transversal, they intercept equal lengths on every transversal.*



* For practical applications, see p. 289.

Given $AB \parallel CD \parallel EF \parallel GH$, and $AC = CE = EG$.

To prove $BD = DF = FH$.

Proof. Draw BO, DP, $FR \parallel AG$.

Then $ACOB$, $CEPD$, and $EGRF$ are ▱s. (136)

$\therefore BO = AC,\ DP = CE,\ FR = EG.$ (141)

But as $AC = CE = EG$. (Hyp.)

$BO = DP = FR.$ (Ax. 1)

$\angle 1 = \angle 2 = \angle 3.$ (106)

$\angle 4 = \angle 5 = \angle 6.$ (106)

$\therefore \triangle BOD \cong \triangle DPF \cong \triangle FRH.$ (s. a. a. = s. a. a.)

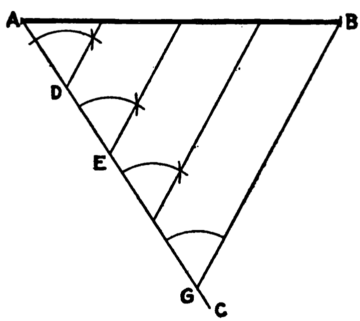
$\therefore BD = DF = FH.$ (Hom. parts of congruent △s.)

Q. E. D.

153. Cor. A parallel to one side of a triangle, bisecting another side, bisects the third side also.

Proposition XXXIX. Problem

154. *To divide a given line into any number of equal parts.*



Given line AB.

Required to divide AB into n equal parts.

Construction. From A draw any line AC.

On AC lay off any segment, as AD, n times in succession, as AD, DE, etc. Join the last point of division, G, with B.

Through the other points of division on AG draw parallels to GB.

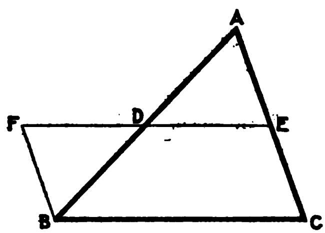
These lines divide AB into n equal parts.

[The proof is left to the student.]

Ex. 369. A line bisecting one non-parallel side of a trapezoid, and parallel to the base, bisects the other non-parallel side.

Proposition XL. Theorem

155. *A line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*



Given $\triangle ABC$, with $AD = DB$, $AE = EC$.

To prove 1° $DE \parallel BC$.

2° $DE = \frac{1}{2} BC$.

Proof. Produce ED by its own length to F. Draw FB.

$AD = DB$, (Hyp.)

$ED = DF$, (Con.)

and $\angle ADE = \angle BDF$. (53)

$\therefore \triangle ADE \cong \triangle BDF$. (s.a.s. = s.a.s.)

$\therefore AE = BF$ and $\angle A = \angle DBF$. (hom. parts etc.)

$\therefore AC \parallel FB$. (93)

But $AE = EC$. (Hyp.)

$\therefore BF = EC$. (Ax. 1.)

$\therefore BCEF$ is a $\square$. (147)

$\therefore DE \parallel BC$, and $DE = \frac{1}{2}(FE) = \frac{1}{2}(BC)$ (141)

Q. E. D.

Ex. 370. If the mid-points of the three sides of a triangle are joined, four equal triangles are formed.

Ex. 371. A line joining the mid-points of two adjacent sides of a quadrilateral is equal and parallel to the line joining the mid-points of the other two.

Ex. 372. The lines joining the mid-points of the sides of any quadrilateral, taken in order, inclose a parallelogram.

Ex. 373. If the mid-points of the arms of an isosceles triangle are joined to the mid-points of the base, an equilateral parallelogram is formed.

Ex. 374. If in $\triangle ABC$, the medians AD and BE meet in O, the line joining the mid-points of AO and BO is equal to DE.

Ex. 375. In quadrilateral $ABCD$, the line connecting the mid-points of AD and AC is equal to the line joining the mid-points of BD and BC.

Ex. 376. If the points A' and B' are lying respectively in the sides AC and BC of $\triangle ABC$, $CA' = \frac{1}{4}(CA)$ and $CB' = \frac{1}{4}(CB)$, then $A'B' = \frac{1}{4}(AB)$.

* **Ex. 377.** The median of a trapezoid is parallel to the base, and equal to half the sum of the bases.

Ex. 378. Given three fixed points A, B, and C, not in a straight line, construct a triangle the mid-points of whose sides are A, B, and C.

[See practical problems, p. 290, No. 26.]

156. Def. An **equiangular polygon** is a polygon all of whose angles are equal. An **equilateral polygon** is a polygon all of whose sides are equal.

A polygon of five sides is called a **pentagon**; one of six sides, a **hexagon**; seven sides, a **heptagon**; eight sides, an **octagon**; ten sides, a **decagon**; twelve sides, a **dodecagon**.

All the polygons discussed are understood to be **convex**, *i.e.* such that no side produced will enter the polygon.

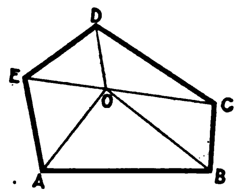
157. Polygons that are mutually equiangular and equilateral are **congruent**, for they can be made to coincide.

Ex. 379. Draw two mutually equiangular quadrilaterals that are not mutually equilateral.

Ex. 380. Draw two mutually equilateral quadrilaterals that are not mutually equiangular.

PROPOSITION XLI. THEOREM

158. *The sum of all the angles of a polygon of n sides is equal to (n − 2) straight angles.*



Given $ABCDE \cdots$, a polygon of n sides.

To prove $\angle A + \angle B + \angle C + \cdots = (n-2)$ st. $\angle$s.

Proof. Connect any point within, as O, with all the vertices. There will be n $\triangle$s thus formed.

Since the sum of the angles of each $\triangle = 1$ st. $\angle$, (110)

The sum of the $\angle$s of all $\triangle$s $= n$ st. $\angle$s.

But the sum of all $\angle$s about $O = 2$ st. $\angle$s. (52)

$\therefore \angle A + \angle B + \angle C + \cdots = (n-2)$ st. $\angle$s.

159. COR. Each angle of an equiangular polygon of n sides equals $\dfrac{n-2}{n}$ straight angles.

Ex. 381. How many straight angles are in the sum of the angles of a polygon of 9 sides? of 12 sides? of 100 sides?

Ex. 382. How many right angles are in a polygon of 20 sides? of 41 sides? of 200 sides? of 1000 sides?

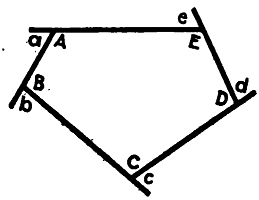
Ex. 383. How many degrees are in the sum of the angles of a polygon of 4 sides? of 5 sides? of 6 sides?

Ex. 384. How many sides has a polygon the sum of whose angles is 1000 st. $\angle$s? 200 rt. $\angle$s? 24 rt. $\angle$s? 720°?

Ex. 385. How many degrees are in each angle of an equiangular quadrilateral? pentagon? hexagon? decagon?

PROPOSITION XLII. THEOREM

160. *If the sides of any polygon be successively produced, forming one exterior angle at each vertex, the sum of these angles is equal to two straight angles.*



Given *ABCDE*, a polygon of n sides, with the ext. $\angle$s a, b, c, etc.

To prove $\angle a + \angle b + \angle c + \cdots = 2$ st. $\angle$s.

Proof. $\angle A + \angle a = 1$ st. $\angle$. (50)

$\angle B + \angle b = 1$ st. $\angle$, etc. (50)

$\therefore (\angle A + \angle B + \cdots) + (\angle a + \angle b + \cdots) = n$ st. $\angle$s. (Ax. 2)

But $(\angle A + \angle B + \cdots) = (n-2)$ st. $\angle$s. (158)

$\therefore \angle a + \angle b + \angle c \cdots = 2$ st. $\angle$s. (Ax. 3)

Ex. 386. How many degrees are in each exterior angle of an equiangular polygon of 10 sides? of 9 sides? of 36 sides? of 72 sides?

Ex. 387. How many sides has a polygon each of whose exterior angles equals 30°? one right angle? 60°? 45°?

Ex. 388. How many sides has a polygon each of whose interior angles equals 160°? 179°? 135°? $\frac{4}{3}$ rt. $\angle$s?

Ex. 389. How many sides has an equiangular polygon, three of whose exterior angles are together equal to 90°?

Ex. 390. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

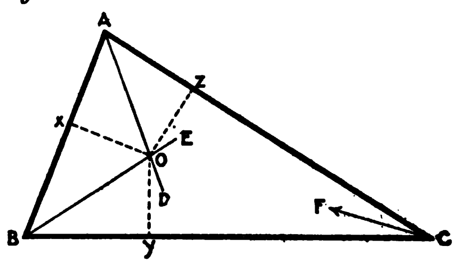
Ex. 391. How many sides has a polygon, the sum of whose interior angles equals twice the sum of its exterior angles?

Ex. 392. How many sides has a polygon each of whose interior angles equals eight times the adjacent exterior angle?

161. Def. Three or more lines are **concurrent** if they meet in a common point.

PROPOSITION XLIII. THEOREM

162. *The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the sides of the triangle.*



Given $\triangle ABC$, and AD, BE, and CF, the bisectors of $\angle s\ A$, B, and C respectively.

To prove (*a*) AD, BE, and CF meet in a point, as O.
(*b*) O is equidistant from AB, BC, and CA.

Proof. Since AD and BE cannot be $\parallel$, (107)
AD and BE meet in a point, as O.

From O draw OX, OY, and OZ respectively $\perp$ AB, BC, and CA.

$OX = OZ$. (125)

$OX = OY$. (125)

$\therefore OZ = OY$. (Ax. 1)

$\therefore O$ lies in CF or its prolongation. (126)

$\therefore AD$, BE, and CF meet in point O, and O is equidistant from AB, BC, and CA. Q.E.D.

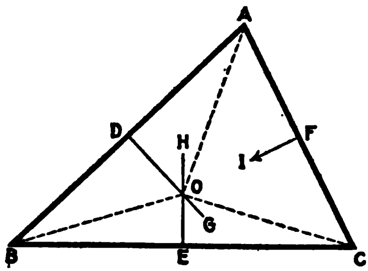
Ex. 393. Find by a construction a point equidistant from the three sides of an obtuse triangle.

Ex. 394. Prove that the bisectors of two exterior angles and the bisector of the third interior angle of a triangle meet in a point.

Ex. 395. Construct a point equidistant from the sides AB, BC, and CD of parallelogram $ABCD$.

PROPOSITION XLIV. THEOREM

163. *The perpendiculars erected at the mid-points of the sides of a triangle are concurrent in a point which is equidistant from the vertices of the triangle.*



Given $\triangle ABC$, and DG, EH, and FI, the perpendicular-bisectors of AB, BC, and CA respectively.

To prove (*a*) DG, EH, and FI meet in a common point. (*b*) The point of intersection is equidistant from A, B, and C.

Proof. DG and EH intersect at a point, as O. (102)

Draw OA, OB, and OC.

$OA = OB$. (123)

$OB = OC$. (123)

$\therefore OA = OB$. (Ax. 1.)

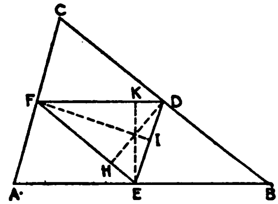
$\therefore O$ lies in the $\perp$ bisector of AC. (81)

Or DG, EH, and FI meet in O; and O is equidistant from A, B, and C. Q. E. D.

Ex. 396. Construct a point O equidistant from three vertices of a quadrilateral.

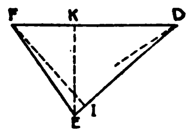
Ex. 397. Construct a circle which passes through the three vertices of a given triangle.

Ex. 398. If *EK*, *FI*, and *DH* are perpendicular bisectors of the sides of triangle *ABC*, what kind of lines are they in regard to triangle *DFE*?



Ex. 399. Must the altitudes of triangle *FDE* (preceding exercise) meet in a point? For what reason?

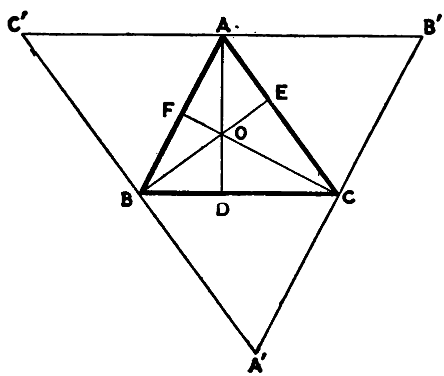
Ex. 400. Construct a triangle *ABC* so that the altitudes of *FED* become perpendicular bisectors of triangle *ABC*.



Ex. 401. What fact relating to the altitudes of any triangle follows from the preceding proposition?

Proposition XLV. Theorem

164. *The three altitudes of a triangle (produced if necessary) are concurrent.*



Given $\triangle ABC$ and the altitudes AD, BE, and CF.

To prove AD, BE, and CF meet in a common point.

Proof. Through the vertices of ABC, draw $B'C' \parallel BC$, $A'C' \parallel AC$, and $A'B' \parallel AB$.

Since $AD \perp BC$, (Hyp.)

$AD \perp B'C'$. (105)

But $ABCB'$ and $ACBC'$ are parallelograms. (136)

$\therefore AB' = BC$, and $AC' = BC$. (141)

$\therefore AB' = AC'$. (Ax. 1.)

Or AD is the $\perp$ bisector of $B'C'$.

In like manner it follows that

BE is the $\perp$ bisector of $A'C'$,

and CF is the $\perp$ bisector of $A'B'$.

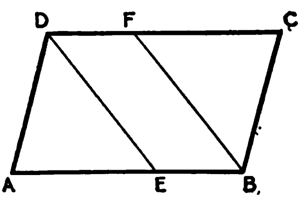
$\therefore AD$, BE, and CF are concurrent. (163)

Q. E. D.

ANALYSIS OF THEOREMS

165. The demonstration of a theorem is most frequently discovered by the so-called **analytic method**. An analysis considers the various *"means"* * by which a conclusion can be proved, and thereby reduces the original proposition to another, simpler one. This reduction is continued until a known fact is obtained. A number of concrete examples will explain the true character of this method.

166. THEOREM. *The bisectors of the opposite angles of a rhomboid are parallel.*



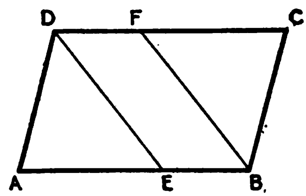
Given DE and BF, the bisectors of two angles of the rhomboid $ABCD$.

To prove $DE \parallel BF$.

* The principal "means" are the directions given under the heading "method," but in the absence of a specific method any theorem or axiom may be used.

Analysis. 1. The means for proving that two lines are parallel are:

(*a*) A parallelogram.
(*b*) Equal alternate interior angles.
(*c*) Corresponding angles.
(*d*) Supplementary interior angles, etc.



Let us select any one of these methods, *e.g.* 1 (*a*), *i.e.*
to prove $EBFD$ is a parallelogram.

2. The means for proving that a quadrilateral is a ▱ are:
(*a*) Opposite sides are equal.
(*b*) Two opposite sides are equal and parallel, etc.

Again we select any one, *e.g.* 2 (*a*), *i.e.*
to prove $ED = BF$, $EB = DF$.

3. The means for proving the equality of lines is usually a pair of congruent triangles, *i.e.*
to prove $\triangle AED \cong \triangle BCF$.

4. The congruence of the two triangles is easily established. Hence we obtain the following demonstration:

Proof.	$AD = BC$.	(141)
	$\angle A = \angle C$.	(141)
	$\angle ADE = \angle FBC$.	(Ax. 8.)
	$\therefore \triangle ADE \cong \triangle FBC$.	(a. s. a. = a. s. a.)
	$\therefore ED = BF$, and $AE = FC$.	(hom. parts, etc.)
But	$AB = DC$.	(141)
	$\therefore DF = EB$.	(Ax. 3.)
	$\therefore DEBF$ is a ▱.	(146)
	$\therefore ED \parallel BF$.	(136) Q. E. D.

The above demonstration is not the only one nor the shortest one. Each of the possibilities indicated under 1 (*a*), 1 (*b*), 1 (*c*), 1 (*d*), 2 (*a*), 2 (*b*), will furnish one or more proofs.

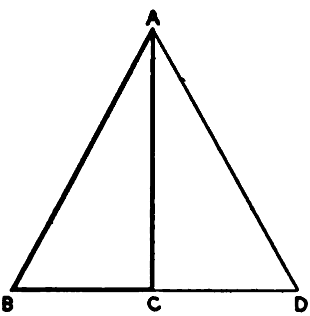
Ex. 402. Prove the above proposition by means of 2 (*b*).

Ex. 403. Prove the above proposition by means of 1 (*b*).

Ex. 404. Prove the above proposition by means of 1 (*c*).

167. When in an analysis new lines or angles are drawn, the student should — before proceeding with the regular analysis — determine whether thereby any new equalities of lines or of angles are produced. This is illustrated by the following

THEOREM. *If one acute angle of a right triangle equals* 60° *the arm forming this angle is one half the hypotenuse.*



Given $\triangle ABC$, with $\angle C = 90°$, and $\angle B = 60°$.

To prove $AB = 2\,(BC)$

Analysis. 1. The usual method of proving that one line is twice as large as another requires the doubling of the smaller line.

$\therefore$ produce BC by its own length to D and

prove $BD = BA$.

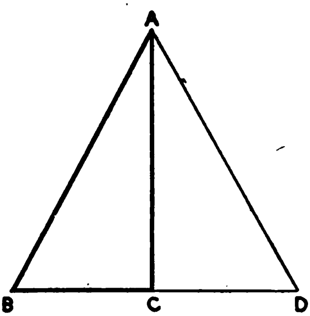
Before proceeding determine any obvious equalities produced by the drawing of new lines. It is easily seen that $\triangle ACD \cong \triangle ACB$ (s. a. s. = s. a. s.), and that hence

$$\angle D = \angle B = 60°,$$
$$\angle CAD = \angle CAB = 30°.$$

2. To prove $BD = BC$ we may use either the method of congruent triangles, or demonstrate in $\triangle BAD$ the equality of the two angles opposite BD and BC.

The former method (which is more difficult) is left to the student. The latter requires the proof for the equality

$$\angle D = \angle BAD.$$



3. Obviously $\angle BAD = 60°$, and since $\angle D = 60°$, the proof can be easily established.

Ex. 405. If in triangle ABC, $\angle A = 60°$, $B = 30°$, and $AB = 12$ inches, find the length of AC.

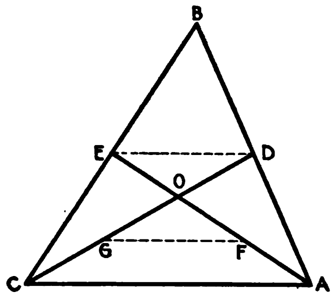
Ex. 406. If in triangle ABC, $AB = 10$ inches, $\angle A = 60°$, $\angle C = 90°$, and CD is the altitude upon AB, find the length of AD.

Ex. 407. The base angles A and B of isosceles triangle ABC equal 15°, and $BC = 8$ inches. Find the length of AD.

Ex. 408. In quadrilateral $ABCD$, $AB = BC = 6$ inches, $\angle A = 120°$, $\angle B = 60°$, and $\angle C = 150°$. Find the length of AD.

Ex. 409. DC, the upper base of trapezoid $ABCD$ equals 14 inches, and $AB = BC$. Find the length of AB, if $\angle A = 90°$, and $\angle B = 60°$.

168. THEOREM. *If a median of a triangle is intersected by another one, the segment of the median between the point of intersection and the vertex is twice its other segment.*



Given the medians DC and AE of $\triangle ABC$ meeting in O.

To prove $AO = 2\,(OE)$; and $CO = 2\,(OD)$.

Analysis. 1. The means of proving that one line equals twice another are:

$$\left.\begin{array}{l}(a)\ \text{Bisect the greater}\\(b)\ \text{Double the smaller}\end{array}\right\}. \qquad (143)$$

Each method furnishes a proof. Select 1 (a). Bisect AO and CO in F and G respectively, and prove $FO = OE$, $GO = DO$.

2. The means of proving the equality of lines is usually a pair of congruent triangles, *i.e.*

to prove (a) $\triangle DOE \cong \triangle FOG$,

or (b) $\triangle DOF \cong \triangle OGE$.

Either pair may be used: *e.g.* 2 (a).

3. Draw DE and FG.

$$\left.\begin{array}{l}DE = \frac{1}{2}AC \text{ and } \parallel AC\\FG = \frac{1}{2}AC \text{ and } \parallel AC\end{array}\right\} \text{why ?}$$

The congruence of the triangles may now be easily established. Hence,

Proof. DE is parallel and equal to $\frac{1}{2}AC$. (155)

FG is parallel and equal to $\frac{1}{2}AC$. (155)

$\therefore DE$ is parallel and equal to FG. (92 and Ax. 1)

$\therefore \angle EDO = \angle OGF$, (104)

and $\angle DOE = \angle FOG$, (53)

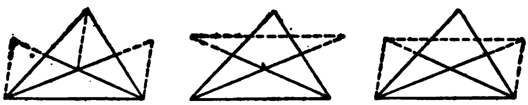
$\therefore \triangle DOE \cong \triangle FOG$, (s. a. a. = s. a. a.)

$\therefore FO = OE$; $GO = OD$. (hom. parts, etc.)

$\therefore AO = 2(OE)$; $OC = 2(OD)$. Q. E. D.

Ex. 410. Demonstrate the proposition by proving the equality of $\triangle DOF$ and $\triangle EOG$. (Draw OB.)

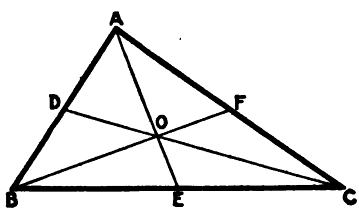
Ex. 411. Prove the proposition by showing that $DFGE$ is a parallelogram.



Ex. 412. Prove the proposition by means of 1(*b*) of analysis (three methods).

PROPOSITION XLVI. THEOREM

169. *The medians of a triangle are concurrent.*



Given AE, BF, CD, the medians of $\triangle ABC$.

To prove AE, BF, and CD are concurrent.

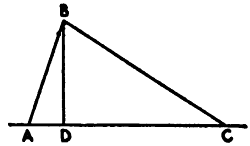
Proof. Trisect EA, and let O be the first point of division.

BF passes through O. (168)

CD passes through O. (168)

$\therefore AE$, BF, and CD are concurrent. Q. E. D.

Ex. 413. If two medians of a triangle are equal, the triangle is isosceles.



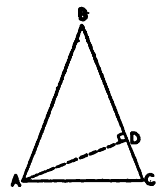
Ex. 414. Two observers, at A and at C, on opposite sides of a balloon, B, measure the angles

$$BAC = 75^\circ \text{ and } BCA = 30^\circ.$$

Find BD, the height of the balloon, if $AC = 8$ miles, and the balloon is directly above a point D in line AC.

[See applied problems 27–31, p. 290.]

Ex. 415. Given the vertex angle B of an isosceles triangle, required $\angle DAC$, formed by the base and the altitude upon one arm, in terms of B.



Solution. Let $\angle B = m^\circ$.

Then $\angle A + \angle C = 180^\circ - m^\circ$. (Why?)

But $\angle A = \angle C$. (Why?)

$\therefore 2\,(\angle C) = 180^\circ - m^\circ$. (Why?)

$\therefore \angle C = 90^\circ - \dfrac{m^\circ}{2}$. (Why?)

But $\angle DAC = 90^\circ - \angle C$. (Why?)

$\therefore \angle DAC = 90^\circ - \left(90^\circ - \dfrac{m^\circ}{2}\right)$. (Why?)

Or $\angle DAC = \dfrac{m^\circ}{2} = \dfrac{B}{2}$.

Therefore the angle formed by the base and the altitude upon an arm of an isosceles triangle equals one half the vertex angle.

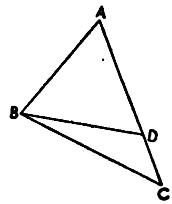
170. Method XI. Geometrical propositions may be demonstrated by algebraical computation.

At this stage of the work, the above method may be employed to establish relations between two or more angles. In case of two angles, express the value of one in algebraic form (*e.g.* m°) and from this try to derive the value of the other.

If a relation between three or more angles has to be demonstrated, express the values of all of them excepting one by algebraic symbols (as m°, n°, p°, etc.), and try to discover the value of the remaining angle. After some practice the symbols which designate the angle, as $\angle A$ or $\angle B$, may be used to express their value.

If the student is unable to find the value of the last angle when all others are given in algebraic form, he should at first attack the problem for numerical values of these angles and then try to solve the general problem.

Ex. 416. Given $\triangle ABC$, and point D in AC, so that $AD = AB$. Prove $\angle DBC = \frac{1}{2}(\angle B - \angle C)$.



Proof. Let $\angle B = m°$, and $\angle C = n°$.

Then $\angle A = 180° - m° - n°$.

But $\angle ABD + \angle BDA = 180° - \angle A$.

I.e. $\angle ABD + \angle BDA = 180° - (180° - m° - n°)$

$= m° + n°$.

But $\angle ABD = \angle ADB$.

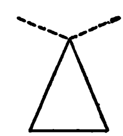
$\therefore 2(\angle ABD) = m° + n°$.

$\therefore \angle ABD = \frac{m°}{2} + \frac{n°}{2}$.

But
$$\angle DBC = \angle B - \angle ABD$$
$$= m° - \left(\frac{m°}{2} + \frac{n°}{2}\right)$$
$$= \frac{m°}{2} - \frac{n°}{2}$$
$$= \frac{B}{2} - \frac{C}{2}$$

or
$$\angle DBC = \tfrac{1}{2}(B - C).$$
Q.E.D.

Ex. 417. Perpendiculars drawn at the vertex of an isosceles triangle upon the arms inclose an angle equal to twice either base angle.



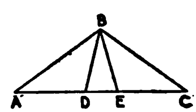
Ex. 418. An exterior angle formed by the prolongation of the base of an isosceles triangle equals 90° increased by one half the vertex angle.

Ex. 419. The sum of the two exterior angles of a right triangle which are formed by the prolongations of the hypotenuse equals 270°.

Ex. 420. If in the base AC of an isosceles $\triangle ABC$, 2 points D and E be taken such that

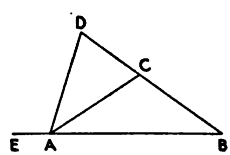
$$AB = AE \text{ and } BC = CD,$$

prove $\angle DBE = \angle A$.

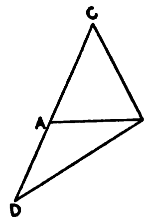


Ex. 421. In the diagram opposite, if $AD = AC = CB$, and DB and EB are st. lines, then

$$\angle EAD = 3(\angle B).$$



Ex. 422. If the arm CA of isosceles triangle ABC be produced so that AD equals the base AB, then $\angle C = 180^\circ - 4(\angle D)$.



Ex. 423. If the base AB of isosceles triangle ABC is produced to D, and CD is drawn, then

$$\angle BCD = \angle A - \angle D.$$

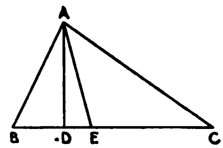
Ex. 424. If in isosceles triangle ABC, one end (B) of the base AB is connected with a point D in the opposite arm, then

$$\angle A = \tfrac{1}{2}(\angle CDB - \angle CBD).$$

Ex. 425. If the side AC of triangle ABC is produced to D so that $CD = CB$, then

$$\angle ABD = 90^\circ + \tfrac{1}{2}(\angle ABC - \angle A).$$

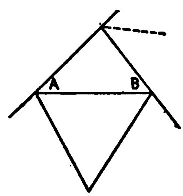
Ex. 426. If AD is the altitude and AE the bisector of the angle BAC of the triangle ABC, prove $\angle DAE = \tfrac{1}{2}(\angle B - \angle C)$.



Ex. 427. The sum of three angles of a quadrilateral, diminished by the fourth exterior angle, is equal to a straight angle.

Ex. 428. The bisectors of two exterior angles of a triangle include an angle equal to one half the third exterior angle.

HINT. Express every angle in terms of $\angle A$ and $\angle B$.



MISCELLANEOUS EXERCISES

Ex. 429. What are the different tests by which to determine the congruence of triangles?

Ex. 430. State some of the properties of all triangles.

Ex. 431. State some of the special properties of isosceles triangles.

Ex. 432. State some of the properties of all parallelograms.

Ex. 433. What additional properties does a rhombus have?

Ex. 434. The bisectors of supplementary adjacent angles are perpendicular to each other.

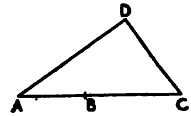
Ex. 435. If the bisectors of two adjacent angles are perpendicular to each other, the exterior sides of these angles form a straight line (*i.e.* they form a straight angle).

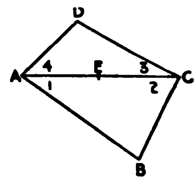
Ex. 436. The bisectors of vertical angles are in a straight line.

Ex. 437. In the diagram opposite, if

$$\angle A + \angle C + \angle D = 180°,$$

then ABC is a straight line.





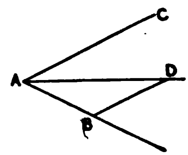
Ex. 438. In the annexed diagram, if

$$\angle 1 + \angle 2 + \angle B = \angle 3 + \angle 4 + \angle D,$$

then AEC is a straight line.

Ex. 439. The altitude upon the hypotenuse of a right triangle divides the figure into two triangles which are mutually equiangular.

Ex. 440. If through any point D on the bisector of an angle A a parallel be drawn to one of the sides, to meet the other side in B, then $AB = BD$.



Ex. 441. If from a point in the bisector of an angle, lines are drawn parallel to the sides of the angle, either a square or a rhombus is formed.

Ex. 442. If the vertex angles of two isosceles triangles are supplementary, the base angles are complementary.

Ex. 443. If an arm of an isosceles triangle is produced by its own length through the vertex, and the end of the prolongation is joined to the nearest end of the base, the line joining is perpendicular to the base.

Ex. 444. Homologous medians of equal triangles are equal.

Ex. 445. Homologous altitudes of equal triangles are equal.

Ex. 446. Two isosceles triangles are equal if the vertex angle and the altitude upon an arm of the one are respectively equal to the vertex angle and the homologous altitude of the other.

Ex. 447. If in the pentagon $ABCDE$,

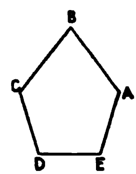
$$AB = BC,$$

$$AE = CD,$$

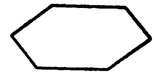
and $\angle A = \angle C,$

then $BE = BD,$

and $\angle E = \angle D.$

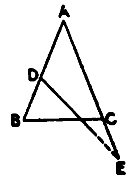


Ex. 448. Two equilateral triangles are equal if the altitude of one equals the altitude of the other.



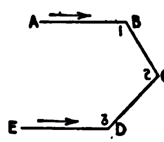
Ex. 449. If the opposite sides of a hexagon are parallel, and two of opposite sides are equal, all opposite sides are equal.

Ex. 450. If from the ends of the base BC of an isosceles triangle ABC, equal parts, BD and CE, be laid off on one arm and the prolongation of the other, the line joining D and E is bisected by the base.



Ex. 451. If two lines are intersected by a transversal, and the bisectors of the interior angles on the same side of the transversal are perpendicular to each other, these lines are parallel.

Ex. 452. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.



Ex. 453. In the diagram given here, if $AB \parallel ED$,

$$\angle 1 + \angle 2 + \angle 3 = 4 \text{ rt. } \angle s.$$

Ex. 454. State and prove the converse of the preceding proposition.

Ex. 455. Find the number of diagonals in a polygon of 5 sides; of 8 sides; of 10 sides; of n sides.

Ex. 456. How many sides has a polygon, the sum of whose interior angles equals three times the sum of the exterior angles? (*I.e.* one ext. $\angle$ at each vertex.)

Ex. 457. How many sides has a polygon the sum of whose interior angles is equal to n times the sum of the exterior angles?

Ex. 458. How many sides has a polygon, the sum of whose interior angles is equal to three times the sum of the angles of a hexagon?

Ex. 459. How many sides has an equiangular polygon whose exterior angle equals the interior angle of an equilateral triangle?

Ex. 460. If the upper base of an isosceles trapezoid is equal to one of the arms, the diagonals bisect the angles at the lower base.

Ex. 461. If a perpendicular be dropped from the vertex to the base of a triangle, each segment of the base will be shorter than the adjacent side of the triangle.

Ex. 462. If the vertices of a triangle lie in the sides of another triangle, the perimeter of the first is less than the perimeter of the second.

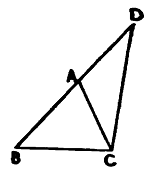
* **Ex. 463.** The perpendiculars from two vertices of a triangle upon the median drawn from the third vertex are equal.

Ex. 464. The lines joining the mid-points of the sides of a rectangle, taken in order, inclose a rhombus.

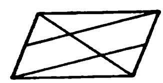
Ex. 465. The lines joining the mid-points of opposite sides of any quadrilateral bisect each other. (Ex. 372.)

Ex. 466. The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals determine the vertices of a parallelogram. (Ex. 375.)

Ex. 467. A line from the vertex of an isosceles triangle to any point in the base is shorter than either arm.

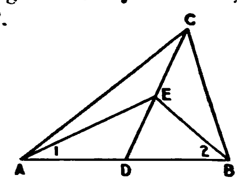


Ex. 468. If in the triangle ABC $AB > AC$, and D is a point in the prolongation of BA, then $DB > DC$.



Ex. 469. Lines joining the mid-points of two opposite sides of a parallelogram to the ends of a diagonal trisect the other diagonal.

* **Ex. 470.** If a point D in a side BC of triangle ABC is joined to A, and $AC = BC$, $AB = AD = DC$, then $\angle C = 36°$.



* **Ex. 471.** If any point E in the median CE is joined to A and B and $\angle B > \angle A$, prove that $\angle 2 > \angle 1$ (annexed diagram).

Ex. 472. From a given point without a line, to draw a line forming with the given line an angle equal to half a right angle.

Ex. 473. From a given point without a line, to draw a line forming with the given line an angle of 60°. How many such lines can be drawn?

Ex. 474. From a given point without a line to draw a line, making a given angle with the given line.

Ex. 475. Construct a line terminating in the sides of a given angle and equal and parallel to a given line.

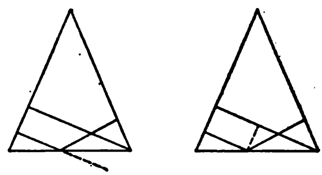
Ex. 476. The diagonals of an isosceles trapezoid are equal.

* **Ex. 477.** If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

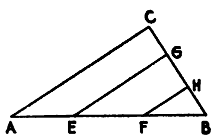
171. METHOD XII. **In order to prove that the sum of two lines, a and b, equals a third line, c, either**

(a) Construct the sum of a and b, and prove the line so obtained is equal to c, or

(b) Lay off a (or b) on c, and prove that the line representing the difference equals b or (a).



Ex. 478. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the arms is equal to the altitude upon one of the arms.



Ex. 479. In $\triangle ABC$, if

$$AE = BF$$

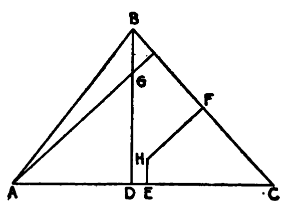
and $$AC \parallel EG \parallel FH,$$

prove that $$EG + FH = AC.$$

Ex. 480. If through a point D in the base AB of isosceles triangle ABC, parallels are drawn to the arms meeting the arms in E and F respectively, then

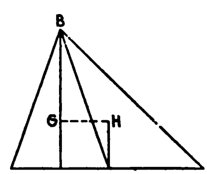
$$DF + DE = AC.$$

Ex. 481. The sum of the three perpendiculars dropped from any point of an equilateral triangle upon the sides is constant, and equal to the altitude of the triangle. (Ex. 478.)



* **Ex. 482.** If the altitude BD of $\triangle ABC$ is intersected by another altitude in G, and EH and HF are perpendicular-bisectors, prove

$$BG = 2(HE), \text{ and } AG = 2(HF).$$



Ex. 483. The line joining the point of intersection of the altitudes of a triangle and the point of intersection of the three perpendicular-bisectors cuts off one third of the corresponding median. (Ex. 482.)

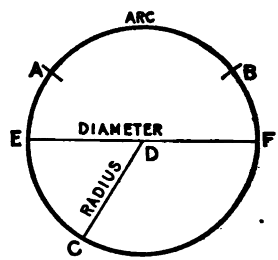
* **Ex. 484.** The points of intersection of the altitudes, medians, and perpendicular-bisectors of a triangle lie in a straight line.

BOOK II

THE CIRCLE — CONSTRUCTIONS

172. Def. A **circle** is a plane closed curve, all of whose points are equally distant from a fixed point in the plane, as *ABC* (37).

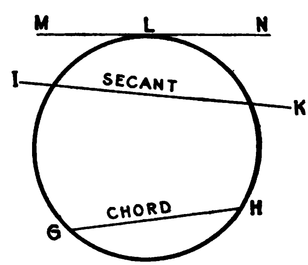
173. Def. The fixed point in the plane (*D*) is the **center**. The length of the circle is called the **circumference**. An **arc** is any portion of the circumference, as *AB*. A **semicircumference** is half of the circumference. A **minor arc** is an arc less than a semicircumference; a **major arc** is an arc greater than a semicircumference.



The word *arc* taken alone generally signifies a minor arc.

174. Def. A **radius** is a line joining the center to any point in the circle, as *DC*. A **diameter** is a straight line through the center terminated at each end by the circle, as *EF*.

175. Def. A **chord** is a straight line joining any two points in the circumference, as *GH*. A **secant** is a straight line that intersects the circle in two points, as *IK*. A **tangent** is a straight line that touches the circle at one point only, and does not intersect it if produced, as *MN*. This point (*L*) is called the *point of contact* or *point of tangency*.



176. Def. A **central angle** is an angle formed by two radii, as $\angle EDC$.

An angle is said to *intercept* any arc that is cut off by its sides, and this arc is said to *subtend* the angle. A chord that joins the ends of an arc is said to *subtend* this arc; thus chord GH subtends the arc GH.

177. Def. Circles having the *same center* are called **concentric.**

PRELIMINARY THEOREMS

178. All radii of the same circle are equal. (By definition.)

179. A point is within, on, or without a circumference, according as the distance from the center is less than, equal to, or greater than a radius.

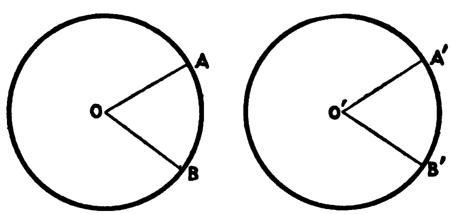
180. All diameters of the same circle or of equal circles are equal.

181. Two circles are equal if their radii are equal. (Prove by superposition.)

182. A diameter bisects the circumference. (Prove by superposition.)

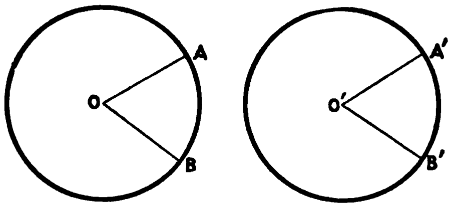
Proposition I. Theorem

183. *In the same circle or in equal circles, equal central angles intercept equal arcs; and, conversely, equal arcs subtend equal central angles.*



I. **Given** in equal ⓢ, O and O',

$$\angle O = \angle O'.$$



To prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Proof. Place the circle whose center is O on the circle whose center is O' so that OA coincides with OA'. (19)

Then OB takes the direction OB $(\angle O = \angle O')$

B coincides with B', $(OB = O'B')$

$\therefore \overset{\frown}{AB}$ coincides with $\overset{\frown}{AB'}$ Otherwise the radius of one ⊙ would be greater than that of the other.

$\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$. Q. E. D.

II. CONVERSELY. I. **Given** in equal ⓢ, O and O',

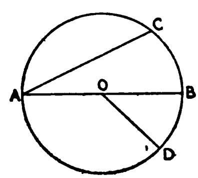
$$\overset{\frown}{AB} = \overset{\frown}{A'B'}.$$

To prove $\angle O = \angle O'$.

HINT. By superposition.

184. COR. In the same or in equal circles, the greater of two unequal central angles intercepts the greater arc, and conversely.

185. METHOD XIII. **The equality of arcs in the same circle or equal circles can be demonstrated by the equality of the subtended central angles.**



Ex. 485. If AB and CD are diameters of the same circle, arc AC = arc BD.

Ex. 486. If, in the diagram opposite, AB is a diameter, OD a radius, AC a chord, and $\angle BOD = 2(\angle A)$, then $\overset{\frown}{BD} = \overset{\frown}{BC}$.

Ex. 487. If a diameter AB bisects the angle A, which is formed by two chords AC and AD, then $\widehat{BC} = \widehat{BD}$.

Ex. 488. If from a point A in a circumference a chord AB and a diameter AC are drawn, a radius parallel to AB bisects $\widehat{BC}$.

Ex. 489. If through a point equidistant from two points in the circumference a radius is drawn, the arc between the two points is bisected.

Ex. 490. If a secant is parallel to a diameter, the lines intercept equal arcs on the circumference.

Ex. 491. Any two parallel secants intercept equal arcs on a circumference. (Ex. 490.)

Ex. 492. If the perpendiculars drawn from a point in the circumference upon two radii are equal, the point bisects the arc intercepted by the two radii.

Ex. 493. If the line joining the mid-points of two radii is equal to the line joining the mid-points of two other radii, the radii intercept equal arcs respectively.

Ex. 494. Divide a circumference into four equal parts.

Ex. 495. Divide a circumference into eight equal parts.

Ex. 496. Divide a circumference into six equal parts.

186. The circumference of a circle may be divided into 360 equal parts called *degrees*. A degree may be subdivided into 60 equal parts called *minutes*, and similarly a minute may be divided into 60 *seconds*.

An angle is *measured by* an arc, if both angle and arc contain the same number of degrees.

Ex. 497. Prove that a central angle is measured by its intercepted arc if the central angle equals

(*a*) $90°$. (*b*) $30°$. (*c*) $1°$. (*d*) $7°$.

(*e*) $m°$, where m is an integer. (*f*) $\left(\frac{1}{9}\right)^{\circ}$.

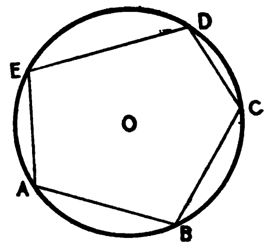
(*g*) $\left(\frac{1}{n}\right)^{\circ}$, when n is an integer.

(*h*) any common fraction $\left(e.g.\ \frac{m}{n}\text{, when } m \text{ and } n \text{ are integers}\right)$.

Ex. 498. Construct an arc of (*a*) $45°$, (*b*) $60°$, (*c*) $150°$.

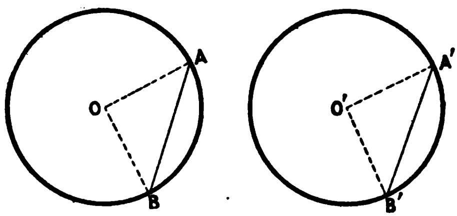
187. Def. A polygon is **inscribed in a circle,** if all its vertices are in the circumference; as *ABCDE*. The circle is then said to be **circumscribed about the polygon.**

The center of the circumscribed circle is called the **circumcenter** of the polygon.



Proposition II. Theorem

188. *In the same circle, or in equal circles, equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.*



I. **Given** in equal ⓢ, O and O',

$$\overset{\frown}{AB} = \overset{\frown}{A'B'}.$$

To prove $AB = A'B'$.

Proof

STATEMENTS	REASONS
Draw OA, OB, OA', and OB'.	
$\angle AOB = \angle A'O'B'$.	In equal ⓢ, equal arcs subtend equal central ∠s.
$AO = A'O'$	Radii of equal ⓢ.
$BO = B'O'$.	Radii of equal ⓢ.
$\therefore \triangle ABO \cong \triangle A'B'O'$.	s. a. s. = s. a. s.
$\therefore$ chord AB = chord $A'B'$.	Homologous parts of congruent △s.

Q. E. D.

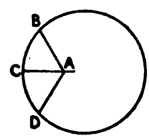
II. CONVERSELY. **Given** in equal $\circledS$, O and O',

$$AB = A'B'.$$

To prove $\widehat{AB} = \widehat{A'B'}$.

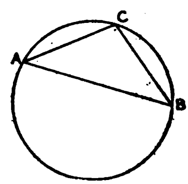
HINT. Draw radii OA, OB, $O'A'$, $O'B'$, and prove the congruence of $\triangle$ OAB and $O'A'B$.

189. METHOD XIV. **The equality of arcs, in the same circle or equal circles, may also be demonstrated by the equality of the subtending chords, and vice versa.**



Ex. 499. In the annexed diagram, if $\angle BAC = \angle DAC$ and $AB = AD$, then $\widehat{BC} = \widehat{CD}$.

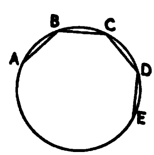
Ex. 500. In the same diagram, if $\angle BAC = \angle DAC$ and $\angle\ CBA = \angle\ CDA$ (not drawn in the figure), then $\widehat{CB} = \widehat{CD}$.



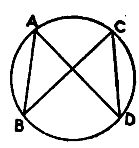
Ex. 501. If $\triangle ABC$ is inscribed in a circle and $\angle A = \angle B$, prove that $\widehat{AC} = \widehat{BC}$.

Ex. 502. If two chords bisect each other, the arcs intercepted by the sides of a pair of vertical angles are equal.

Ex. 503. If two chords bisect each other, they are diameters.



Ex. 504



Exs. 505 and 506

Ex. 504. If chords AB, BC, CD, DE are equal, then chords AC, BD, CE are equal.

Ex. 505. If in the diagram opposite, $AB = CD$, then $BC = AD$.

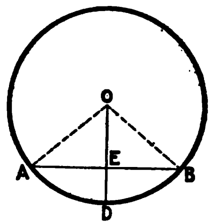
Ex. 506. If in the same diagram two intersecting chords, AD and BC, are equal, then $AB = CD$.

Ex. 507. The diagonals of an equilateral pentagon inscribed in a circle are equal.

Ex. 508. The radii drawn to the vertices of an inscribed equilateral hexagon divide the figure into six equilateral triangles.

PROPOSITION III. THEOREM

190. *A radius perpendicular to a chord bisects the chord and the subtended arc.*



Given in $\odot O$, the radius $OD \perp AB$.

To prove $AE = EB$.

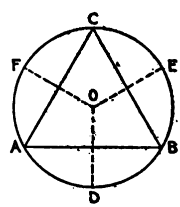
$\overset{\frown}{AD} = \overset{\frown}{DB}$.

HINT. Draw radii AO and OB.

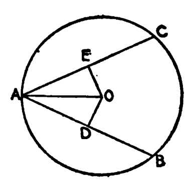
What is the usual means of proving:

(1) The equality of lines?
(2) The equality of arcs?

191. COR. A perpendicular-bisector of a chord passes through the center of the circle.



Ex. 509. In the diagram opposite, if the radii OD, OE, and OF are respectively perpendicular to the sides of the equilateral triangle ABC, then $\overset{\frown}{AD} = \overset{\frown}{DB} = \overset{\frown}{BE} = \overset{\frown}{EC} = \overset{\frown}{CF} = \overset{\frown}{FA}$.



Ex. 510. If, in the diagram opposite, $AB = AC$, $OD \perp AB$, and $OE \perp AC$, prove that $\triangle ADO \cong \triangle AEO$.

Ex. 511. In the same diagram, if $OD \perp AB$, $OE \perp AC$, and $OD = OE$, prove that $AB = AC$.

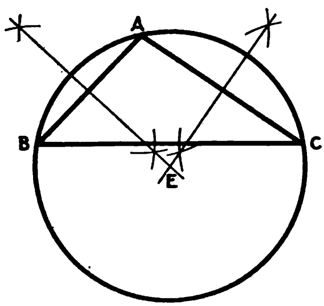
Ex. 512. If two chords are equal, the perpendiculars from the center upon the chords are equal.

Ex. 513. If from a point without a circle two equal lines are drawn to a circumference, the bisector of the angle they form passes through the center of the circle.

Ex. 514. Two points, each equidistant from the ends of a chord, determine a line passing through the center of the circle.

Proposition IV. Problem

192. *To circumscribe a circle about a given triangle.*



Given △ *ABC*.

Required to circumscribe a circle about △ *ABC*.

Construction. Draw the perpendicular bisectors of the sides *AC* and *AB*. (83)

They will intersect at some point *E*. (102)

From *E* as a center, with a radius equal to *EA*, describe a circle *ABC*.

ABC is the required circle.

Proof. *E* is equidistant from *A*, *B*, and *C*. (163)

∴ A circle drawn from *E* as a center with a radius equal to *EA* will pass through *A*, *B*, and *C*. Q. E. D.

193. Cor. 1. Three points not in a straight line determine a circle.

194. Cor. 2. A circle cannot be drawn through three points which lie in the same straight line.

195. Cor. 3. **A straight line cannot intersect a circle in more than two points.**

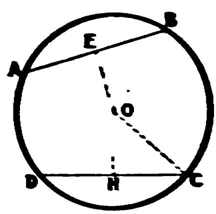
196. Cor. 4. **Two circles cannot meet in more than two points.**

Ex. 515. To find the center of a given circle.

Ex. 516. To bisect a given arc.

Proposition V. Theorem

197. ***In the same circle, or in equal circles, equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.***



I. **Given** in $\odot$ $ABCD$

chord $AB =$ chord CD, $OE \perp AB$, $OH \perp CD$.

To prove $OE = OH$.

Hint. What is the means of proving the equality of lines?

II. Conversely. **Given** in $\odot$ $ABCD$,

$OE = OH$, $OE \perp AB$, $OH \perp CD$.

To prove $AB = CD$

The proof is similar to the above.

198. Method XV. **The equality of two chords in the same circle, or in equal circles, is usually established by means of equal distances from the center or equal subtended arcs.**

Ex. 517. If the perpendiculars from the center upon the sides of an inscribed polygon are equal, the polygon is equilateral.

Ex. 518. If from any point in the circumference two chords are drawn making equal angles with the radius to the point, these chords are equal.

Ex. 519. If through any point in a radius two chords are drawn making equal angles with the radius, these chords are equal.

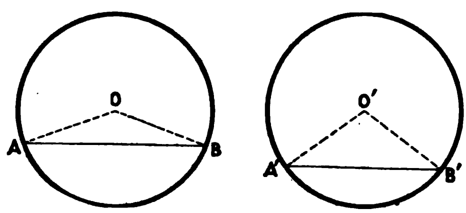
Ex. 520. A line joining the point of intersection of two equal chords to the center bisects the angle formed by the chords.

Ex. 521. In a given circle, to draw a chord equal and parallel to a given chord.

Ex. 522. In a given circle, to draw a chord equal to a given chord, and parallel to a given line.

Proposition VI. Theorem

199. *In the same circle, or in equal circles, the greater of two minor arcs is subtended by a greater chord; and conversely, the greater chord subtends the greater arc.*



I. **Given** in equal ⓢ, O and O',

$$\overset{\frown}{AB} > \overset{\frown}{A'B'}.$$

To prove $AB > A'B'$.

Proof. Draw radii OA, OB, $O'A'$, $O'B'$.

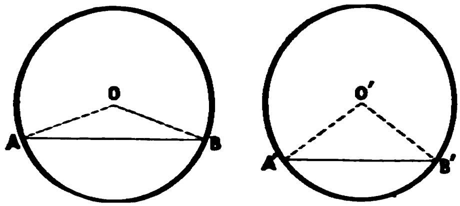
In $\triangle$ AOB and $A'O'B'$,

$$\angle O > \angle O', \quad (184)$$

but $AO = A'O'$ and $BO = B'O'$ (181)

$$\therefore AB > A'B'. \quad (132)$$

Q. E. D.



II. Conversely. **Given** in equal $\circledS$, O and O',

$$AB > A'B'.$$

To prove $\qquad \overset{\frown}{AB} > \overset{\frown}{A'B'}.$

Proof. Draw radii OA, OB, $O'A'$, $O'B'$.

In $\triangle$ AOB and $A'O'B'$,

$$AO = A'O',\ BO = B'O', \qquad (181)$$

$$AB > A'B'. \qquad \text{(Hyp.)}$$

$$\therefore \angle AOB > \angle A'O'B'. \qquad (133)$$

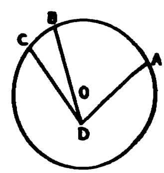
$$\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}. \qquad (184)$$

Q. E. D.

Ex. 523. If ABC is any triangle inscribed in a circle and $\angle A > \angle B$, then $\overset{\frown}{BC} > \overset{\frown}{AC}$.

Ex. 524. State and prove the converse of the preceding proposition.

Ex. 525. In the diagram opposite, if



$$AD = DC,$$

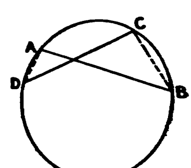
and $\qquad \angle ADB > \angle BDC,$

then $\qquad \overset{\frown}{AB} > \overset{\frown}{BC}.$

Ex. 526. In the same diagram, if $AD = DC$, and $\overset{\frown}{AB} > \overset{\frown}{BC}$, then $\angle ADB > \angle BDC$.

Ex. 527. If upon a radius two perpendicular chords are drawn, the one nearer the center is the greater chord.

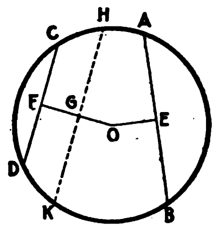
Ex. 528. In the diagram opposite, if $AB > CD$, and arcs ACB and DAC are minor arcs, prove that $CB > AD$.



Ex. 529. In the same diagram, if $CB > AD$, and arcs DAC and ACB are minor arcs, prove that $AB > CD$.

Proposition VII

200. *In the same circle, or in equal circles, chords unequally distant from the center are unequal, the nearer one being the greater, and conversely.*



Given in ⊙ $ABDC$, $OE \perp$ chord AB, $OF \perp$ chord CD, and

$$OF > OE.$$

To prove $AB > CD$.

Proof. On OF lay off $OG = OE$.

Through G draw chord $HK \perp OF$.

Then $HK = AB$. (197)

Since $OG < OF$, G lies between O and F.

$\therefore \overset{\frown}{HK} > \overset{\frown}{CD}$. (Ax. 11)

$\therefore HK > CD$. (199)

$\therefore AB > CD$.

Q.E.D.

II. Conversely. **Given** $AB > CD$.

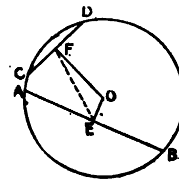
To prove $OF > OE$.

Prove by the indirect method.

201. Cor. The diameter is greater than any other chord.

202. Method XVI. **The inequality of chords, in the same circle or in equal circles, is usually established by means of unequal distances from the center or by means of unequal arcs.**

Ex. 530. The perpendicular from the center of a circle to a side of an inscribed equilateral hexagon is less than the perpendicular from the center to the side of an inscribed equilateral octagon.



Ex. 531. In the diagram opposite, if $EO \perp AB$, $OF \perp CD$, and $\angle OEF > \angle OFE$, then $AB > CD$.

Ex. 532. If $\triangle ABC$ is inscribed in a circle and $\angle B > \angle C$, then the perpendicular from the center upon AB is greater than the perpendicular from the center upon AC.

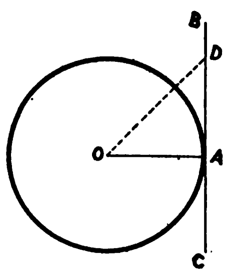
Ex. 533. The shortest chord which can be drawn through a point within a circle is perpendicular to the radius drawn through the point.

Ex. 534. Two chords drawn from a point in the circumference are unequal if they make unequal angles with the radius drawn from that point.

* **Ex. 535.** Two chords drawn through an interior point are unequal if they make unequal angles with the radius drawn through that point.

Proposition VIII. Theorem

203. *A straight line perpendicular to a radius at its outer extremity is a tangent to the circle.*



Given in $\odot O$,

radius $OA \perp BC$ at A.

To prove BC is a tangent.

Proof. Join any other point D in BC to O.

$OD > OA$. (131)

$\therefore D$ lies without the circumference. (179)

$\therefore BC$ is a tangent to $\odot O$. (175) Q. E. D.

204. Cor. 1. **A tangent is perpendicular to the radius drawn to the point of contact.**

205. Cor. 2. **A perpendicular to a tangent at the point of contact passes through the center of the circle.**

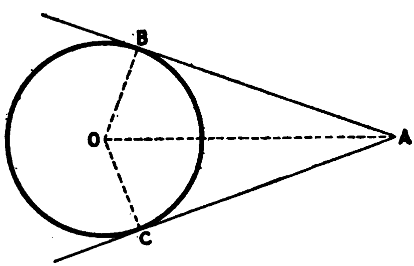
206. Cor. 3. **A perpendicular from the center to a tangent meets it at the point of contact.**

207. Cor. 4. **At a given point of contact there can be one tangent only.**

208. Def. The **length of tangent** drawn from a point to a circle is the length of the line from this point to the point of contact; as AB (Prop. IX).

Proposition IX. Theorem

209. *The tangents drawn to a circle from a point without are equal.*



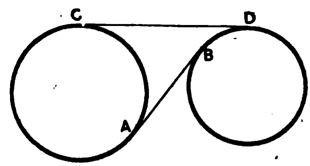
Given tangents AB and AC to circle O.

To prove $AB = AC$.

Hint. What is the means of proving the equality of lines?

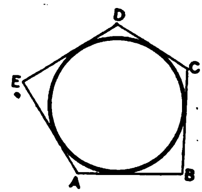
210. Cor. **The tangents drawn to a circle from a point without make equal angles with the line joining the point to the center** (*i.e* $\angle\ OAB = \angle\ OAC$).

211. Def. A common tangent to two circles is called an **internal tangent** if it lies between the two circles, as AB. Otherwise it is called a **common external tangent,** as CD.



The **length** of a common tangent is the length of the segment between the **points of contact.**

212. Def. A polygon is **circumscribed about a circle** if all its sides are tangent to the circle; as $ABCDE$. The circle is then said to be inscribed in the polygon.



The center of the inscribed circle is called the *incenter* of the polygon.

Ex. 536. The common internal tangents of two circles are equal.

Ex. 537. The common external tangents of two circles are equal.

Ex. 538. A chord forms equal angles with the tangents drawn at its ends.

Ex. 539. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Ex. 540. If hexagon $ABCDEF$ is circumscribed about a circle, $AB + CD + EF = BC + DE + FA$.

Ex. 541. The sum of the arms of a right triangle circumscribed about a circle is equal to the hypotenuse increased by the diameter of the circle.

Ex. 542. If two tangents make an angle of 60°, the chord joining the points of contact equals either tangent.

***Ex. 543.** Triangle ABC is circumscribed about a circle, which touches the sides AB, BC, and CA in X, Y, and Z, respectively. Find the lengths of AX, BY, and CZ if $AB = 3$, $BC = 4$, and $CA = 5$.

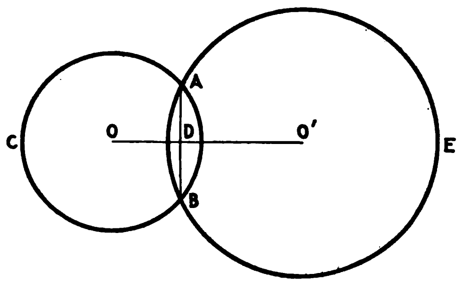
[See practical applications, p. 291.]

213. Def. The **line of centers** is the line joining the centers of two circles.

Thus OO' (diagram for Prop. X) is the line of centers of circles O and O'.

PROPOSITION X. THEOREM

214. *If two circles intersect, the line of centers bisects their common chord at right angles.*



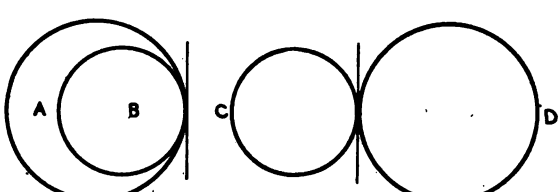
Given circumferences ACB and AEB intersecting at A and B.

To prove OO', their line of centers, is the perpendicular-bisector of AB.

Proof. O and O' are each equally distant from A and B. (178)

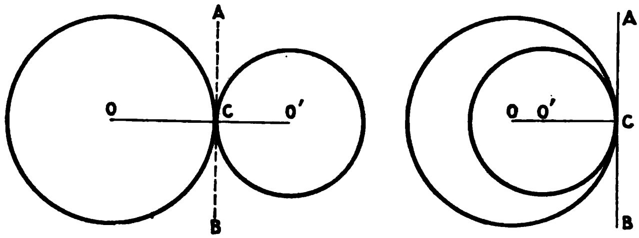
$\therefore$ OO' is the perpendicular-bisector of AB. (80) Q. E. D.

215. DEF. Two circles are **tangent to each other** if both are tangent to the same straight line at the same point. They are tangent internally or externally, according as one circle lies within or without the other. Thus, circles A and B are tangent internally, and circles C and D are tangent externally.



PROPOSITION XI. THEOREM

216. *If two circles are tangent to each other, their line of centers passes through the point of contact.*



Given OO' is the line of centers of circles O and O', that are tangent at C.

To prove OO' passes through C.

Proof. Draw common tangent AB at C. At the point of contact, C, draw a perpendicular to the common tangent.

This perpendicular passes through O and O'. (205)

$\therefore$ OO' passes through C. Q. E. D.

Ex. 544. What are the relative positions of two circles, if the line of centers is

(*a*) Greater than the sum of the radii?
(*b*) Equal to the sum of the radii?
(*c*) Less than the sum but greater than the difference of the radii?
(*d*) Equal to the difference of the radii?
(*e*) Less than the difference of the radii?
(*f*) Equal to zero?

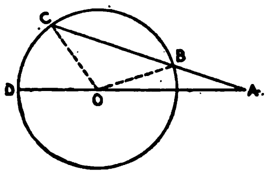
Ex. 545. If in the diagram of the preceding proposition (XI) tangents are drawn from A to circles O and O', these tangents are equal.

Ex. 546. If a secant intersects two concentric circles, its segments, intercepted by the two circumferences, are equal.

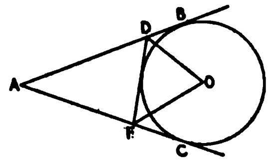
Ex. 547. Two parallel chords, drawn from the extremities of a diameter, are equal.

Ex. 548. In the diagram given here, if the radius OB is equal to AB, prove $\angle COD = 3\angle A$.

Ex. 549. If from A the tangents AB and AC are drawn to a given



Ex. 548

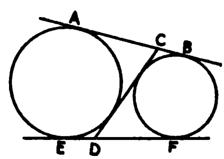


Ex. 549

circle O, and a third tangent intersects AB and AC in D and F,

Prove (1) $AD + DF + AF = 2\,AB$.

(2) $\angle O = \text{rt.} \angle - \dfrac{A}{2}$.



* **Ex. 550.** The two common external tangents of two circles intercept on a common internal tangent a segment, CD, equal to the external tangent, AB.

MEASUREMENT

217. To **measure** a quantity is to find how many times it contains another quantity of the same kind, called a **unit of measurement.**

218. Def. The **numerical measure** of a quantity is the number that expresses how many times the quantity contains the unit.

Thus a line is measured by finding the number of yards or inches that it contains. If the result is 16 inches, then 16 is the numerical measure of the line.

219. Def. A number is **rational** if it can be expressed as a common fraction, proper or improper.

Obviously this definition includes all integers.

Thus $\frac{3}{7}$, 9, $2\frac{3}{4}$, 11.4 are rational numbers. If m and n are integers, all rational numbers can be represented in the form $\dfrac{m}{n}$.

220. Def. A number is **irrational** if it cannot be expressed by a common fraction, proper or improper.*

Thus $\sqrt{2}$ cannot be a fraction, for if $\sqrt{2}$ were equal to $\frac{m}{n}$ (where m and n are integers that have no common factor), then $2 = \frac{m \cdot m}{n \cdot n}$, which is impossible since m and n have no common factors. Hence $\sqrt{2}$ is an irrational number. Other irrational numbers are $\frac{\sqrt{7}}{5}$, $\sqrt[3]{9}$, $\sqrt[5]{11}$, etc.

221. In the following chapters we shall designate as *successive approximate values* the decimals that we obtain if we calculate the values of irrational numbers to one, two, three, etc., decimal places.

Thus the successive approximate values of $\sqrt{2}$ are:

$$1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad \text{etc.}$$

Since the true value of $\sqrt{2}$ lies between 1.4 and 1.5, the first approximation differs from the true value by less than .1. Similarly the second approximation differs from the true value by less than .01, the third by less than .001, etc. It is evident that by continuing the decimal this difference may be diminished to less than one millionth or one billionth or less than any assigned value, however small.

222. Irrational numbers cannot be expressed exactly in figures, but approximate values can be found that differ from the true value by less than any assigned value, however small.

223. Two irrational numbers are called *equal* if *all* their successive approximate values are respectively equal.

Obviously two numbers that satisfy this condition cannot differ by any value, however small.

* The definition of "irrational" as given in many texts on elementary algebra refers to a special kind of irrational numbers only, viz. those that can be obtained by evolution. This is the only kind of irrational numbers that can be obtained by the operations of elementary algebra.

Thus, it can be shown that all successive approximate values of $\sqrt{3}$ and $\frac{3}{\sqrt{3}}$ are respectively equal, no matter to how many places of decimals we carry the calculation. Hence we call $\sqrt{3}$ and $\frac{3}{\sqrt{3}}$ equal irrational numbers.

224. Def. The **ratio** of two quantities of the same kind is the quotient of their numerical measures expressed in terms of the same unit.

Thus, the ratio of two quantities, a and b, is $\frac{a}{b}$ or $a \div b$; the ratio of four yards and two yards is $\frac{4}{2}$, or 2. A ratio is used to compare the magnitude of two quantities.

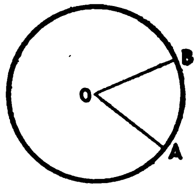
225. If the ratio of two quantities is a rational number, there always exists a third quantity, called a *common measure,* which is contained an integral number of times in each.

Thus, if AB and CD designate the length of two lines expressed in terms of the same unit, and $\frac{AB}{CD}=\frac{5}{9}$, then the ninth part of CD is contained in AB five times. Or if CD be divided into nine equal parts and one of these parts be laid off on AB, as often as possible, then AB contains 5 of these parts. Each of these parts is a common measure of AB and CD.

226. If the ratio of two quantities is an irrational number, it can be shown that they have no common measure. *E.g.* if $\frac{AB}{CD}=\sqrt{7}$, AB and CD cannot have a common measure, for otherwise $\sqrt{7}$ would equal a rational number.

PROPOSITION XII. THEOREM*

227. *A central angle is measured by its intercepted arc.*



Given central $\angle O$ intercepting $\widehat{AB}$.

To prove $\angle O$ and $\widehat{AB}$ are measured by the same number of degrees.

Proof. CASE I. The number of degrees contained in O is rational.

Then the number of degrees in O can be represented by $\frac{m}{n}$, where m and n are integers.

If we construct 360 equal angles about O, each angle would intercept $\frac{1}{360}$ of the circumference. (183)

Or, A central $\angle$ of $1°$ intercepts an arc of $1°$.

If the angle of $1°$ be divided into n equal parts, all intercepted arcs would be equal.

Or, A central $\angle$ of $\left(\frac{1}{n}\right)^{\circ}$ intercepts an arc of $\left(\frac{1}{n}\right)^{\circ}$.

Taking m of these parts, we have

$$\text{A central } \angle \text{ of } \left(\frac{m}{n}\right)^{\circ} \text{ intercepts an arc of } \left(\frac{m}{n}\right)^{\circ}.$$

CASE II. The numbers of degrees contained in O is irrational.

Let the number of degrees contained in O be some irrational number, *e.g.* $\sqrt{2} = 1.41421 \cdots$ and let us consider the successive approximate values of this number.

* Teachers who prefer the proof which is based upon the principle of limits will find the same in the appendix, pp. 279–281.

If O contained 1.4°, $\overset{\frown}{AB}$ contains 1.4°. (Case I)
If O contained 1.41°, $\overset{\frown}{AB}$ contains 1.41°. (Case I)
If O contained 1.414°, $\overset{\frown}{AB}$ contains 1.414°. (Case I)

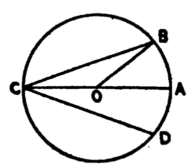
In other words all approximations of the numerical measures of $\angle O$ and of $\overset{\frown}{AB}$ are respectively equal.

$\therefore$ the numerical measures of O and $\overset{\frown}{AB}$ are equal. (223)

Q. E. D.

Note. In the above demonstration a concrete numerical value is assumed for O. It is evident, however, that the proof does not at all depend upon the particular value selected; in other words, that the conclusion follows in like manner for any other numerical value of O. Hence the proof is general. For a more general form of proofs of this type see § 344.

228. Cor. In the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs.

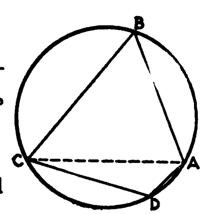


Ex. 551. In the diagram given here, if $\overset{\frown}{AB} = 40°$, how many degrees are (*a*) in central $\angle AOB$? (*b*) in $\angle ACB$?

Ex. 552. In the same diagram if arc $BD = 60°$, how many degrees are in $\angle C$?

[See applied problem, p. 291, Ex. 34.]

229. Def. An **inscribed angle** of a circle is an angle whose vertex lies in the circumference and whose sides are chords, $\angle A$, or $\angle B$.

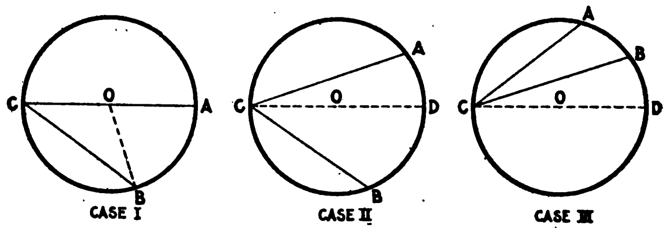


230. Def. A **segment** of a circle is a portion of a circle bounded by an arc and its chord, as ACD.

231. Def. An *angle* is said to be **inscribed in a segment** if its vertex lies in the arc and its sides pass through the extremities of that arc. Thus $\angle D$ is inscribed in segment ACD.

PROPOSITION XIII. THEOREM

232. *An inscribed angle is measured by one half the intercepted arc.*



Given $\angle ACB$ inscribed in $\odot O$.

To prove $\angle ABC$ is measured by $\frac{1}{2}AB$.

Proof. CASE I. One side of the angle is a diameter of the circle.

Draw OB.

Then	$BO = OC$,	(Why?)
and	$\angle C = \angle B$,	(Why?)
	$\angle AOB = \angle C + \angle B$.	(Why?)
	$\therefore \angle AOB = 2\angle C$.	(Why?)
	$\therefore \angle ACB = \frac{1}{2}\angle AOB$.	(Why?)
But	$\angle AOB$ is measured by $\overset{\frown}{AB}$.	
	$\therefore \angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.	Q. E. D.

CASE II. The center lies within the angle.

Draw the diameter CD.

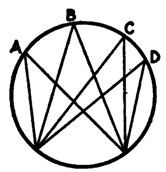
Then	$\angle ACD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.	(Case I.)
and	$\angle BCD$ is measured by $\frac{1}{2}\overset{\frown}{BD}$.	(Case I.)
	$\therefore \angle ACD + \angle BCD$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BD})$.	(Ax. 2.)
Or	$\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.	

CASE III. The center lies without the angle.

Draw the diameter CD.

Then	$\angle ACD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$.	(Case I.)
and	$\angle BCD$ is measured by $\frac{1}{2}\overset{\frown}{BD}$.	(Case I.)
	$\therefore \angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.	(Ax. 3.)

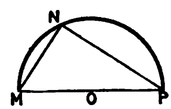
Q. E. D.



233. COR. 1. Angles inscribed in the same segment, or in equal segments, are equal

$$(\angle A = \angle B = \angle C = \angle D).$$

234. COR. 2. An angle inscribed in a semicircle is a right angle ($\angle N = 90°$).



NOTE. The statement "$\angle AOB$ is measured by $\overset{\frown}{AB}$" is a brief form for writing an equation, viz. "the numerical measure of $\angle AOB =$ the numerical measure of $\overset{\frown}{AB}$." Hence, this and similar statements may be treated like equations, *e.g.* the axioms II, III, and VIII may be applied to them.

Another way of treating such statements is to denote the numerical measure of any angle or of any arc by the corresponding italics. Then the above statement would be written:

$$\angle acb = \overset{\frown}{ab}.$$

In this form the proof of Case III is as follows:

$$\angle acd = \tfrac{1}{2}\overset{\frown}{ad}.$$
$$\angle bcd = \tfrac{1}{2}\, bd.$$
$$\therefore \angle acb = \tfrac{1}{2}\, ab.$$

Ex. 553. If in the diagram for Case I, $\angle C = 30°$, how many degrees are in $\overset{\frown}{CB}$?

Ex. 554. If in the same diagram $\overset{\frown}{BC} = 3\,\overset{\frown}{AB}$, find $\angle C$.

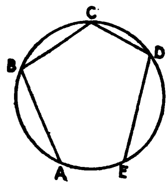
Ex. 555. If in the diagram for Case II $\overset{\frown}{AC}$ is $\frac{1}{4}$, and $\overset{\frown}{BC}$ is $\frac{1}{4}$ of the circumference, find $\angle ACB$, $\angle ACD$.

Ex. 556. If in the diagram for Case III, A is the mid-point of $\overset{\frown}{CD}$, and B is the mid-point of $\overset{\frown}{AD}$, how many degrees are there in $\angle ACB$?

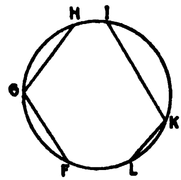
Ex. 557. If a quadrilateral $ABCD$ be inscribed in a circle, and the two diagonals be drawn, find all the angles in the figure, if $\overset{\frown}{AB} = 80°$, $\overset{\frown}{BC} = 110°$, and $\overset{\frown}{CD} = 90°$.

Ex. 558. In the diagram of the preceding exercise, find four pairs of equal angles.

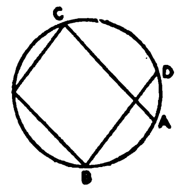
Ex. 559. The opposite angles of an inscribed quadrilateral are supplementary.



Ex. 560



Ex. 561



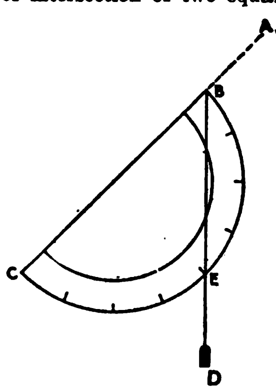
Ex. 562

Ex. 560. In the diagram above, find $\angle B + \angle D$, if $\overset{\frown}{AE} = 40°$.

Ex. 561. In the diagram above, how many degrees are in $\angle G + \angle K$, if $\overset{\frown}{HI} = 20°$, and $\overset{\frown}{FL} = 30°$?

Ex. 562. In the diagram above, how many degrees are in $\angle B + \angle C$, if $\overset{\frown}{DA} = 50°$?

Ex. 563. If through one of the points of intersection of two equal circles a line be drawn to meet the circumferences, the extremities of that line are equidistant from the other point of intersection.

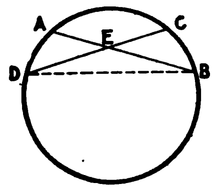


Ex. 564. If a semicircle is placed so that the prolongation of its diameter CB passes through a point A, and a plumb line BD intersects the semicircle at E, the angle of elevation of A is measured by $\frac{1}{2}\overset{\frown}{BE}$.

[For the definitions of angle of elevation and additional practical applications, see problems 34–36, p. 291.]

PROPOSITION XIV. THEOREM

235. *An angle formed by two chords intersecting within the circle is measured by one half the sum of the intercepted arcs.*



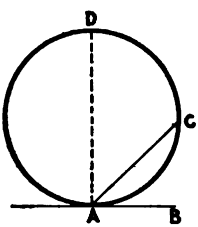
Given two chords AB and CD intersecting in E in $\odot O$.

To prove $\angle AED$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{CB})$.

HINT. **Draw** DB, **and apply Prop. XIII.**

PROPOSITION XV. THEOREM

236. *An angle formed by a tangent to a circle and a chord drawn from the point of contact is measured by half the intercepted arc.*



Given AB a tangent, and AC a chord in $\odot O$.

To prove $\angle CAB$ is measured by $\frac{1}{2}\overset{\frown}{AC}$.

Proof. Draw the diameter AD.

The rt. $\angle DAB$ is measured by $\frac{1}{2}\overset{\frown}{DCA}$.

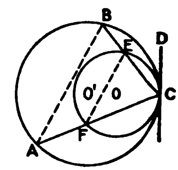
$\angle DAC$ is measured by $\frac{1}{2}\overset{\frown}{DC}$. (232.)

$\therefore \angle CAB$ is measured by $\frac{1}{2}\overset{\frown}{CA}$. (Ax. 3.) Q. E. D.

Ex. 565. If in diagram for Prop. XIV $\widehat{AD} = 60°$, $\widehat{BA} = 140°$, and $\widehat{CB} = 20°$, find $\angle AEC$ and $\angle DAE$.

Ex. 566. If two perpendicular chords intersect within the circle, the sum of a pair of opposite intercepted arcs is equal to a semicircumference.

Ex. 567. Prove Prop. XV by demonstrating the equality of $\angle ADC$ and $\angle CAB$.



Ex. 568. If, in Prop. XV, arc $AC = 2$ arc CD, find $\angle CAB$.

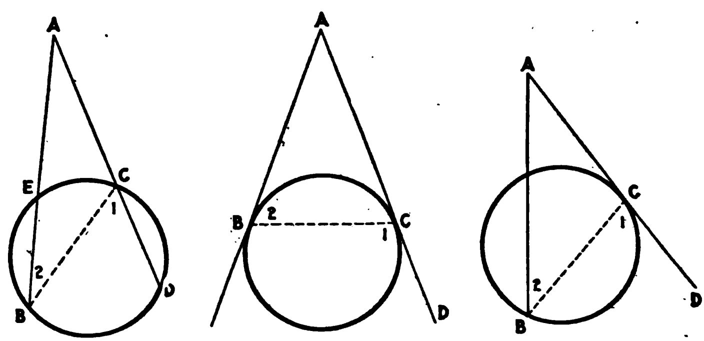
Ex. 569. A chord is parallel to the tangent drawn through the mid-point of the subtended arc.

Ex. 570. In the diagram given here, circles O and O' are tangent to DC at C. Prove that

$$\angle DCB = \angle A = \angle EDC.$$

Proposition XVI. Theorem

237. *An angle formed by two secants, or two tangents, or a tangent and a secant, intersecting without a circle, is measured by half the difference between the intercepted arcs.*



Case I. **Given** AB and AD secants drawn from an external point A to $\odot O$, cutting circle at E and C.

To prove $\angle A$ is measured by $\frac{1}{2}(\widehat{BD} - \widehat{EC})$.

Hint. $\angle A = \angle 1 - \angle 2$.

Similarly for Case II and Case III.

Ex. 571. If the angle formed by two tangents is 60°, how many degrees are in each of the intercepted arcs?

Ex. 572. If in the diagram for Case I, $\widehat{BD} = 100°$ and $\angle A = 20°$, find $\widehat{EC}$.

Ex. 573. If in the same diagram $\widehat{EC} = 60°$ and $\widehat{EB} = \widehat{BD} = \widehat{CD}$, find $\angle A$.

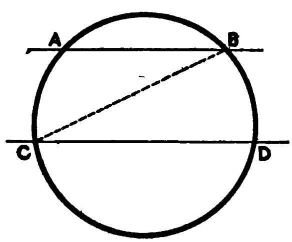
Ex. 574. If an angle formed by a secant and a tangent is 20° and the greater of the intercepted arcs is 90°, how many degrees are in the other intercepted arc?

238. Scholium. If we consider an arc which intersects the sides of an angle as negative when it turns its convex side toward the vertex, and positive when it turns the concave side toward the vertex, or passes through the vertex, Props. XIII, XIV, XV, and XVI may be stated as follows:

239. Theorem. *If the sides of an angle (indefinitely produced) intersect or touch a circumference, the angle is measured by one half the algebraic sum of the intercepted arcs.*

Proposition XVII. Theorem

240. *Parallel secants intercept equal arcs on a circumference.*



Given AB and CD two parallel secants to $\odot O$.

To prove $\widehat{AC} = \widehat{BD}$.

Draw BC and prove by means of Prop. XIII.

K

241. Cor. 1. If a secant is parallel to a tangent, they intercept equal arcs on a circumference.

242. Cor. 2. Two parallel tangents intercept equal arcs on a circumference.

Ex. 575. State and prove the converse of Proposition XVII.

Ex. 576. State and prove the converse of Corollary I.

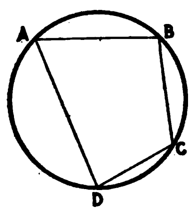
Ex. 577. In the diagram for Prop. XVII, find $\angle ABC$ if $\widehat{AB} = 80°$ and $\widehat{CD} = 120°$.

Ex. 578. The line bisecting an angle formed by a tangent and a chord bisects the intercepted arc.

Ex. 579. If two sides of an inscribed quadrilateral are parallel, the other two sides are equal.

Proposition XVIII. Theorem

243. *The opposite angles of an inscribed quadrilateral are supplementary.*

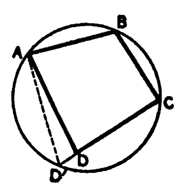


Given $ABCD$ an inscribed quadrilateral.

To prove $\angle A + \angle C = 2$ rt. $\angle$s.

Hint. Find the arcs by which $\angle A$ and $\angle C$ are measured.

Ex. 580. In the diagram for Prop. XVIII, prove that an exterior angle at C is equal to $\angle A$.



***Ex. 581.** If the opposite angles of a quadrilateral are supplementary, its vertices are concyclic, *i.e.* a circumference can be described through them.

Hint. Construct a circle through A, B, and C, and assume that it does not pass through D. Then $\angle D$ would equal D', etc.

Ex. 582. Find the sum of three alternate angles of an inscribed hexagon.

Ex. 583. The corresponding segments of two equal intersecting chords are equal.

Ex. 584. If in the greater of two concentric circles, chords be drawn touching the smaller circle, the chords are equal.

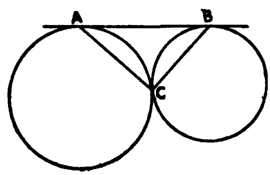
Ex. 585. If two equal chords intersect, the lines joining their ends form an isosceles trapezoid.

Ex. 586. If the bisector of an inscribed angle be produced until it meets the circumference, and through this point of intersection a chord be drawn parallel to one side of the angle, it is equal to the other side.

Ex. 587. A circle constructed on an arm of an isosceles triangle as diameter bisects the base.

Ex. 588. If two circles intersect in A and B and AC and AD are diameters of the two circles, then a line joining C and D passes through B.

Ex. 589. If two circles are tangent to each other and through the point of contact two lines are drawn, each terminating in the two circumferences, the chords joining the ends of these lines are parallel.

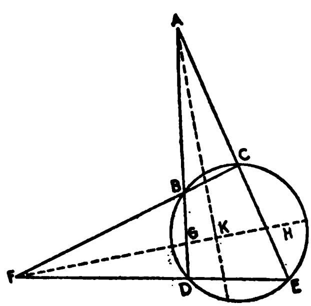


Ex. 590. If two circles are tangent at C and a common exterior tangent touches the circles in A and B, the angle ACB is a right angle.

* **Ex. 591.** If from the extremities of a diameter perpendiculars be drawn upon any chord (produced, if necessary), the feet of the perpendiculars are equidistant from the center.

* **Ex. 592.** If two unequal chords be produced to meet, the secants thus formed are unequal.

Ex. 593. If the opposite sides of an inscribed quadrilateral be produced to meet in A and F, the bisectors of the angles A and F meet at right angles.



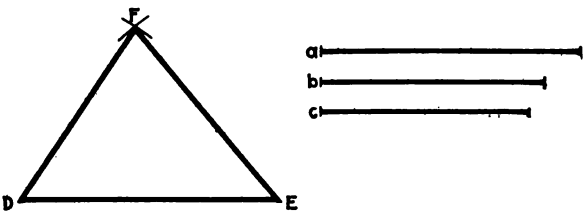
HINT. Prove $\angle BGK = \angle CHK$ (117)

CONSTRUCTIONS

244. Note. In the following examples, we shall denote the given parts of a triangle always in the same manner; the sides by a, b, c, the opposite angles by A, B, and C, the altitudes by h_a, h_b, and h_c, the medians by m_a, m_b, and m_c, and the angle-bisectors by t_a, t_b, and t_c.

Proposition XIX. Problem

245. *To construct a triangle, having given the three sides.*



Given a, b, and c the three sides of a $\triangle$.

Required to construct the triangle.

Construction. Draw $DE = a$.

From E as a center, with a radius equal to b, draw an arc.

From D as a center, with a radius equal to c, draw an arc.

The arcs intersect at F. Draw FE and FD.

$\triangle DEF$ is the required triangle. Q.E.F.

Discussion. The construction is impossible if one side is greater than, or equal to, the sum of the other two.

Ex. 594. Construct an equilateral triangle, having one side given.

Ex. 595. Construct an equilateral triangle whose perimeter equals a given line.

Ex. 596. Construct a quadrilateral, having given the four sides and one diagonal.

PROPOSITION XX. PROBLEM

246. *To construct a triangle, having given two angles and the included side.*

The solution is left to the student.

Ex. 597. Upon a given base, to construct an isosceles right triangle.

Ex. 598. Construct an isosceles triangle, having given the base and a base angle.

Ex. 599. To construct a quadrilateral, having given the four sides and one angle.

Is it possible to solve this exercise by constructing first a side not adjacent to the given angle?

247. REMARK. **The possibility of a solution of a problem depends often upon the proper choice of the part which is drawn first.**

Ex. 600. To construct a rhombus having given the perimeter and one angle.

PROPOSITION XXI. PROBLEM

248. *To construct a triangle, having given two sides and the included angle.*

The solution is left to the student.

Ex. 601. Construct an isosceles triangle, having given an arm and the vertical angle.

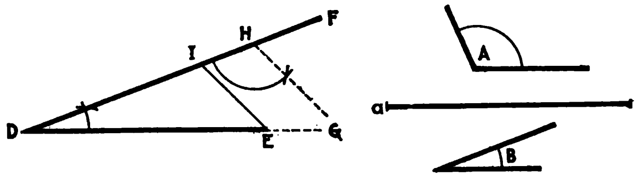
Ex. 602. Construct a right triangle, having given the two arms.

Ex. 603. To construct a quadrilateral ($ABCD$), having given three sides (AB, BC, and CD), the angle formed by the last of these sides and the unknown side ($\angle D$), and the angle formed by the same side and a diagonal ($\angle DCA$).

Ex. 604. To construct a pentagon ($ABCDE$), having given the five sides (AB, BC, CD, DE, and EA) and two angles including one of these sides ($\angle A$ and $\angle B$).

Proposition XXII. Problem

249. *To construct a triangle, having given one side, one adjacent and one opposite angle.*



Given A and B, two angles of a $\triangle$, and a, the side opposite $\angle A$.

Required to construct the triangle.

Construction. Draw $DE = a$.

At D draw $\angle EDF = \angle B$.

At any point, H, in DF, construct $\angle DHG = \angle A$.

Through E, draw a line parallel to HG, meeting DH in I.

$\triangle DIE$ is the required $\triangle$.

Proof.

$DE = a.$	(Con.)
$\angle D = \angle B.$	(Con.)
$\angle DIE = \angle DHG,$	(106)
$\angle A = \angle DHG.$	(Con.)
$\therefore \angle DIE = \angle A.$	(Ax. 1)

Discussion. The construction is impossible if the sum of the given angles is greater than or equal to a straight angle.

Ex. 605. Construct by means of Prop. XXII a right triangle, having given the hypotenuse and an acute angle.

Ex. 606. Find a construction of the same problem which does not depend upon Prop. XXII.

Ex. 607. Construct a right triangle, having given an arm and the opposite angle.

Ex. 608. Construct an equilateral triangle, having given

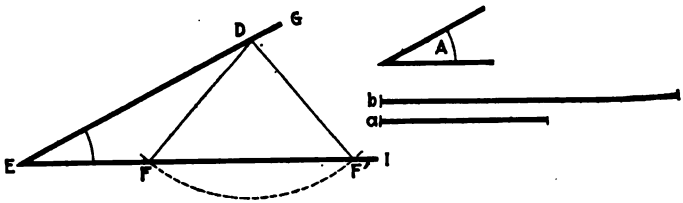
(*a*) the altitude.

(*b*) the radius of the inscribed circle.

(*c*) the radius of the circumscribed circle.

Proposition XXIII. Problem

250. *To construct a triangle, having given two sides and an angle opposite one of them.*



Given a and b, two sides of a $\triangle$, and $\angle A$ opposite side a.

Required to construct the $\triangle$.

Construction. Draw $\angle GEI = \angle A$.

On EG, lay off $ED = b$.

From D as a center with a radius equal to a, draw an arc intersecting EI in F and F'.

Both $\triangle$s EDF and EDF' fulfill the required conditions.

Discussion. If the arc intersects the base twice, there are two solutions, and if it touches the line or intersects once there is one solution. If it does not touch the line, a solution is impossible.

Ex. 609. In the Prop. XXIII, how many solutions are possible, when angle A is obtuse? right? acute?

Ex. 610. Construct a right triangle, having given the hypotenuse and one arm.

Ex. 611. Solve the preceding problem by drawing the hypotenuse first.

251. A triangle may be constructed if the following parts are given:

(1) Three sides.

(2) Two sides and the included angle.

(3) Two angles and the included side.

(4) Two angles and a non-included side.

(5) Two sides and the angle opposite one of them.

To construct a triangle, three independent parts must be given.

Ex. 612. Are the three angles of a triangle three independent parts, and can a triangle be constructed when the three angles are given?

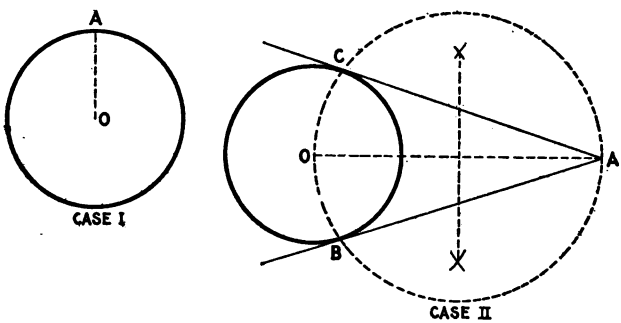
Ex. 613. Is a triangle determined if there is given

(*a*) a, b, m_a (*b*) b, h_a, C?

[See practical problems 38–43, pp. 292 and 293.]

Proposition XXIV. Problem

252. *From a given point, to draw a tangent to a given circle.*



I. When the given point, A, is in the circumference.

Hint. What is the angle formed by a radius and a tangent at its extremity?

II. When the given point, A, is without the circle.

Construction. Join A and O the center of the given circle. On OA as a diameter, construct a circumference intersecting the given circumference in B and C.

Then AC and AB are the required tangents.

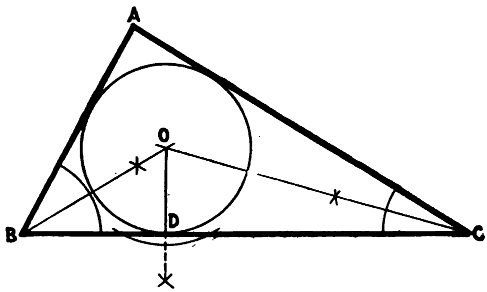
Hint. Show that $\angle s$ ACO and OBA are right angles.

Ex. 614. Construct a line tangent to a given circle and parallel to a given line.

Ex. 615. Construct a line tangent to a given circle and perpendicular to a given line.

Proposition XXV. Problem

253. *To inscribe a circle in a given triangle.*



Given $\triangle ABC$.

Required to inscribe a circle in $\triangle ABC$.

Construction. Construct the bisectors of $\angle B$ and $\angle C$.

These bisectors meet at some point O. (162)

From O draw $OD \perp BC$.

From O as a center, with a radius equal to OD, draw a circle, which is the required one.

[The proof is left to the student.]

254. Def. A circle touching one side of a triangle and the prolongations of the other two sides is an **escribed circle**.

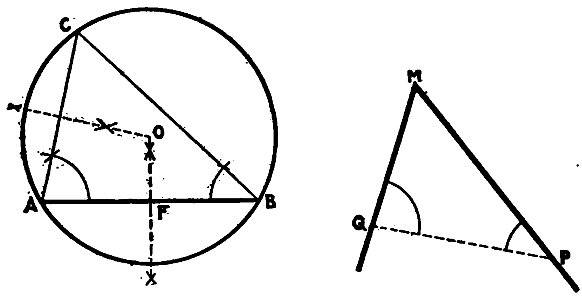
Ex. 616. Construct the three escribed circles of a triangle.

Ex. 617. The bisector of an angle of a triangle meets the circumference of the circumscribed circle in a point which is equidistant from the other two vertices of the triangle and the center of the inscribed circle.

Ex. 618. A circle is inscribed in a triangle whose sides are 9, 8, and 9. Find the distances of the vertices from the points of contact (209).

PROPOSITION XXVI. PROBLEM

255. *Upon a given straight line as chord, to construct a segment of a circle which shall contain an angle equal to a given angle.*



Given line AB and $\angle M$.

Required to construct a segment of a circle on AB as chord which shall contain an angle equal to $\angle M$.

Construction. In the sides of $\angle M$ take any two points P and Q respectively.

Draw PQ.

On AB as a base construct $\triangle ABC$ so that $\angle A = \angle Q$, and $\angle B = \angle P$.

Circumscribe a circle about $\triangle ABC$.

Segment ACB is the required segment. Q. E. F.

Proof $\angle C = \angle M$ (113)

$\therefore$ Any angle inscribed in segment $= \angle M$. (233)

Ex. 619. Given rectangle $ABCD$ with $AB > AD$. To find in AB or its prolongation a point X so that $\angle CXD = 45°$.

Ex. 620. In the diagram for Prop. XXV find the length of AC, if $AB = 6$, $BC = 7$, and $BD = 4$.

[See practical applications, 44 and 45, p. 292.]

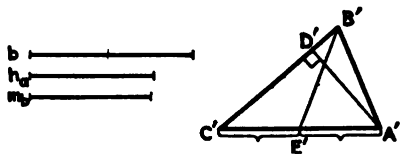
ANALYSIS OF PROBLEMS

256. An **analysis** of a problem is a course of reasoning by which its construction is discovered. Although no rules can be given which apply to all constructions, the method explained in the following exercises may be used in many problems.

Ex. 621. *To construct a triangle having given one side, the corresponding median, and the altitude to another side.*

Given b, one side of triangle, m_b the corresponding median, and h_a the altitude upon another side.

Required to construct the $\triangle$.



Analysis. (1) Suppose $A'B'C'$ were the required triangle.

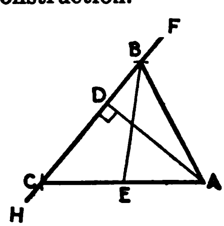
(2) Then we should know $C'A'(=b)$, $C'E'$ and $E'A'\left(=\frac{b}{2}\right)$, $A'D'(=h_a)$, $B'E'(=m_b)$, and $\measuredangle s\ A'D'C'$ and $A'D'B'$ (= rt. $\measuredangle s$).

(The student is advised to mark the known parts or to draw them in a different color from the other lines.)

(3) Examine all triangles in the figure, and see if one can be constructed. The rt. $\triangle A'D'C'$ can be constructed, having given two sides.

(4) Make this triangle the basis of the construction.

Hence,



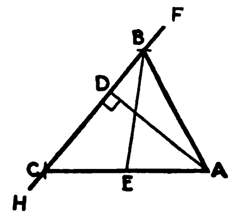
Construction. Draw $DA = h_a$.

At D, draw $FH \perp AD$.

From A as a center, with a radius equal to b, draw an arc intersecting DH in C. Draw AC.

Bisect AC in E, and from E as a center, with a radius equal to m_b describe an arc, meeting FH in B.

ABC is the required triangle.



Proof. $AD = h_a$.

$BE = m_b$.

$C = b$.

AD is an altitude, as

$\angle ADC =$ rt. $\angle$.

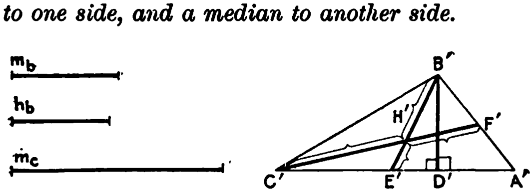
$CE = EA$.

$\therefore EB$ is a median.

257. The following rules express the procedure in a general form:

1. Make a diagram resembling the one required, but not necessarily having the same dimensions.
2. Determine (*a*) all lines, (*b*) all angles, that are directly given, or that can be easily found from the given parts, and mark these parts.
3. Examine all triangles of the diagram until you discover a triangle that can be constructed.
4. Make this triangle the basis of the construction, and try to determine successively all other points of the figure.
5. In case no triangle can be found that can be constructed directly, draw additional lines which will enable you to obtain such a triangle.

Ex. 622. *To construct a triangle, having given the median and altitude to one side, and a median to another side.*



Analysis. 1. Suppose $A'B'C'$ were the required triangle.

2. Then we should know $B'D'\,(= h_b)$, $B'E'\,(= m_b)$, $C'F'\,(= m_c)$,

$E'H'(=\frac{1}{3}m_b)$, $B'H'(=\frac{2}{3}m_b)$, $H'F'(=\frac{1}{3}m_c)$, $H'C'(=\frac{2}{3}m_c)$, and $\angle s\ B'D'E'$ and $B'D'A'(=\text{rt.} \angle s)$.

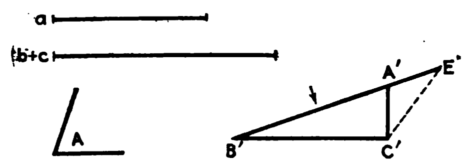
3. Examine all triangles. $B'E'D'$ is the only one that can be constructed.

4. Make the $\triangle B'E'D'$ the basis of the construction.

258. Some problems require the drawing of additional lines (as stated above under 5). In particular:

When a sum or a difference is given, construct such sum or difference in the analysis.

Ex. 623. *To construct a triangle, having given the base, the sum of the other two sides, and the angle included by the two.*



Analysis. 1. Suppose $A'B'C'$ were the required triangle.

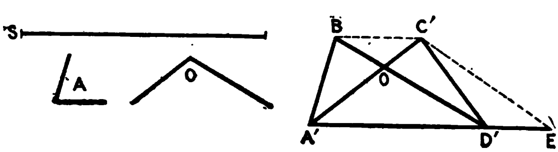
2. Then we should know two parts only.

$\therefore$ produce BA' to E' so that $A'E' = A'C'$.

Draw $E'C'$. We now know $C'B'(=a)$, $B'E'(=b+c)$, $\angle B'A'C'(=A)$, $\angle E'\left(=\frac{A}{2}\right)$, and $\angle A'C'E'\left(=\frac{A}{2}\right)$.

3. Examine all triangles. As $B'C'E'$ can be constructed, make this the basis of the construction.

Ex. 624. *To construct a trapezoid, having given the sum of the bases* (s), *a diagonal* (d), *a base angle* (A), *and the angle formed by the diagonals* (O).



Analysis. 1. Suppose $A'B'C'D'$ were the required trapezoid. Produce $A'D'$ to E' so that $D'E' = B'C'$. Draw $C'E'$.

2. We know $A'E'(=s)$, $A'C'(=d)$, $\angle A'O'D'(=O)$, $\angle B'A'D'$ $(=A)$, $\angle A'C'E'(=A'O'D'=O)$.

3. $\triangle A'C'E'$ can be constructed.

Construct a triangle, having given: (244)

Ex. 625. a, b, m_b.

Ex. 626. a, b, h_b.

Ex. 627. $b, h_a, \angle A$.

Ex. 628. $b, t_c, \angle A$.

Ex. 629. $a, h_b, \angle B$.

Ex. 630. a, b, h_c.

Ex. 631. $a, m_a, \angle B$.

Ex. 632. $\angle C, t_c, b$.

Ex. 633. $t_c, \angle C, \angle B$.

Ex. 634. $\angle A, \angle B, h_a$.

Ex. 635. $\angle A, \angle B, t_c$.

Ex. 636. a, h_a, m_a.

Ex. 637. b, h_a, m_a.

Ex. 638. b, h_a, t_a.

Ex. 639. a, m_b, m_c.

Ex. 640. a, m_a, m_b.

Ex. 641. a, h_b, h_c.

Ex. 642. $h_a, \angle B, \angle C$.

Ex. 643. $m_a, h_a, \angle B$.

Ex 644. $a, b, \angle A + \angle B$.

Ex. 645. $a, \angle B, b + c$.

Ex. 646. $\angle A, \angle C, b - c$.

Ex. 647. Construct an isosceles triangle, having given the base and the vertex angle.

Ex. 648. Construct an isosceles triangle, having given the sum of base and an arm, and a base angle.

Ex. 649. Construct a triangle, having given an angle, an adjacent side, and the difference of the other two sides.

Ex. 650. Construct a triangle, having given the base, the difference of the other two sides, and the angle included by the two.

Construct a right triangle, having given:

Ex. 651. One arm and the altitude upon the hypotenuse.

Ex. 652. The hypotenuse and the difference between the arms.

Ex. 653. The hypotenuse and the sum of the arms.

Ex. 654. Construct a parallelogram, having given the sides and a diagonal.

LOCI

259. **A locus of a point** in a plane is a line or a group of lines, all points of which satisfy a certain condition, satisfied by no other points.

Thus (*a*) Every point in the perpendicular-bisector of a line is equidistant from the ends of a line.

(*b*) No point without the perpendicular-bisector is equidistant from the ends of the line.

∴ The perpendicular-bisector is the *locus* of the point that is equidistant from the ends of the line.

260. The word *locus* (plural *loci*) means "place," and it signifies in geometry the place of a point that moves according to certain conditions.

Thus, if one end of a stretched chord is fixed upon a drawing board, the other end, when moved about, will describe a circle. Hence the circle is the locus of the other end.

Every point of an elevator car moving up or down moves in a straight line, which is perpendicular to the ground. Hence such a straight line is the locus of the moving point.

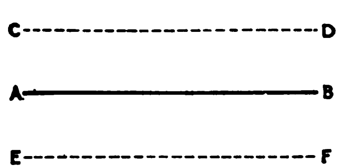
Similarly the locus of a point of the minute hand of a watch is a circle, etc.

261. To prove that a certain line is a locus, we must establish that:

(1) **Every point in the line or group of lines satisfies the given condition.**

(2) **No point without the line or group of lines satisfies the given condition.**

Thus, to find the locus of a point at a distance of $\frac{3}{8}$ inch from AB, it is not sufficient to prove that every point in CD has the proper distance from AB. It is also necessary to show that no point without satisfies the condition, which leads here to the discovery of another part of the locus, EF, located on the other side of AB.



Ex. 655. Draw the following loci without giving proofs:

(*a*) The locus of a point $\frac{3}{4}$ in. from a given point.

(*b*) The locus of a point $\frac{1}{4}$ in. from a given line.

(*c*) The locus of a point equidistant from a pair of parallel lines.

(*d*) The locus of the mid-point of all chords drawn in a given circle and parallel to a given line.

(*e*) The locus of the mid-points of all lines that are parallel to the base of a triangle and terminated by the other sides.

(*f*) The locus of a point equidistant from two given lines.

(*g*) The locus of the end of a line tangent to a given circle and $\frac{1}{2}$ in. long.

(*h*) The locus of the center of a circle whose radius equals $\frac{1}{4}$ in., and that touches a given circle (radius 1 in.) externally.

LOCUS THEOREMS

262. The locus of a point at a given distance from a given point is the circumference described from the point with the given distance as radius.

For (*a*) every point in the circumference has the required distance, while (*b*) a point not in the circumference has not.

263. The locus of a point equidistant from the ends of a given line is the perpendicular-bisector of that line.

For (*a*) every point in the perpendicular-bisector is equidistant. (*b*) No point without is equidistant, since all equidistant points must lie in the perpendicular-bisector (81).

NOTE. In this paragraph the fact (*b*) that no point without satisfies the condition was proved by showing that every point which satisfies the condition must lie in the line. This method can always be employed, and is in some cases more convenient than the regular one.

264. The locus of a point that is at a given distance from a given straight line consists of two lines parallel to the given line at the given distance.

265. The locus of a point equidistant from two given parallel lines is a third parallel, bisecting any line ending in the given parallels.

266. The locus of a point equidistant from two intersecting straight lines consists of the bisectors of the included angles.

Ex. 656. Find the locus of the mid-points of the radii of a given circle.

Ex. 657. Find the locus of the vertex of all right angles whose sides pass through two given points.

Ex. 658. Find the locus of the center of a circle which has a given radius and touches a given line.

Ex. 659. Find the locus of the center of a circle which has a given radius and passes through a given point.

Ex. 660. Find the locus of the center of a circle that passes through two given points.

Ex. 661. Find the locus of the center of a circle that touches two given lines.

Ex. 662. Find the locus of the center of a circle which has a given radius and touches a given circle.

Ex. 663. Find the locus of the center of a circle touching a given line at a given point.

Ex. 664. Find the locus of the center of a circle that touches a given circle in a given point.

Ex. 665. Two vertices, B and C, of a triangle have a fixed position. Find the locus of the third vertex (A) if h_a equals a given line.

Ex. 666. Two vertices (B and C) of a triangle have a fixed position. Find the locus of the third vertex (A), if m_a equals a given line.

Ex. 667. The base of a parallelogram has a fixed position and length, and the adjacent side has a given length. Find the locus of the intersection of the diagonals.

Ex. 668. The base of a rectangle has a fixed position. Find the locus of the intersection of the diagonals.

Ex. 669. Find the locus of the mid-points of all chords that have a given length and are drawn in a given circle.

Ex. 670. Find the locus of the mid-points of all chords that can be drawn from a given point in the circumference.

Ex. 671. The locus of the vertex of a triangle which has a given (fixed) base and a given vertex angle is an arc (constructed according to 255, p. 138).

267. Loci are used to determine the position of a point (or points) that satisfies two conditions. Each condition determines a locus, and the point (or points) of intersection of the two loci is the required point.

In the following exercises, state under what conditions no point, one point, or several points may be found.

Ex. 672. In a given line, AB, find a point at a given distance, d, from a given point C.

Ex. 673. In a given line, AB, find a point at a given distance, d, from a given line CD.

Ex. 674. In a given line, AB, find a point equidistant from two given points, P and Q.

Ex. 675. In a given circumference, find a point at a given distance, d, from a given point C.

Ex. 676. In a given circumference, find a point equidistant from two given parallel lines, CD and EF.

Ex. 677. In a given circumference, find a point equidistant from two given intersecting lines, CD and EF.

Ex. 678. Find a point equidistant from two given intersecting lines, AB and CD, and at a given distance from a given point, E.

Ex. 679. Find a point equidistant from two given intersecting lines, AB and CD, and at a given distance from a given line, EF.

Ex. 680. Find a point equidistant from two given intersecting lines, AB and CD, and equidistant from two given points, E and F.

Ex. 681. Find a point equidistant from two given points, and having a given distance from a given point, E.

Ex. 682. Find a point equidistant from two given points and equidistant from two given parallel lines, EF and GH.

Ex. 683. Find a point equidistant from two given parallel lines and equidistant from two given intersecting lines, EF and GH.

Ex. 684. Find a point at a given distance d, from a given line, AB, and equidistant from two given points, E and F.

Ex. 685. Find a point having a given distance, d, from a given line, AB, and equidistant from two given parallel lines, EF and GH.

To construct a circle having a given radius:

Ex. 686. Touching a given line and passing through a given point.

Ex. 687. Touching two given lines.

Ex. 688. Passing through a point and touching a given circle.

Ex. 689. Touching two given circles.

Ex. 690. Touching a given circle and a given line.

To construct a circle:

Ex. 691. Touching a given line in a given point and passing through another point without.

Ex. 692. Touching a given circle at a given point and passing through another given point without.

Ex. 693. Touching a given line and a given circle at a given point. (HINT. Draw a tangent to the circle at the given point.)

***Ex. 694.** Construct a triangle having given a, h_a, A.

Ex. 695. Construct a triangle having given a, m_a, A.

268. No general method can be given for the solution of exercises; a great many, however, can be solved:

(1) **By a gradual putting together of the given parts.** (247)

(2) **By means of an analysis.**

(3) **By means of loci.**

MISCELLANEOUS EXERCISES

Construct an isosceles triangle, having given:

Ex. 696. The base and the altitude upon an arm.

Ex. 697. The altitude upon the base and the vertex angle.

Ex. 698. The vertex angle and the sum of one arm and the base.

Ex. 699. The perimeter and the base angles.

Construct a right triangle, having given:

Ex. 700. One acute angle and the altitude upon the hypotenuse.

Ex. 701. The altitude upon the hypotenuse and one of the segments of the hypotenuse.

Ex. 702. The sum of the arms and one acute angle.

Ex. 703. To find a point in one side of a triangle which is equidistant from the other two sides.

Ex. 704. Find the locus of the vertex of a right triangle, having a given hypotenuse.

Ex. 705. In one side of a quadrilateral to find a point equidistant from the ends of the opposite side.

Ex. 706. From a point P, in the circumference of a circle, to draw a chord, having a given distance from the center.

Ex. 707. In a given circle, to draw a diameter having a given distance from a given point.

Ex. 708. Through a given point to draw a line, having a given distance from another point.

Ex. 709. Through two given points in a circumference, to draw two equal parallel chords.

Ex. 710. Trisect a given straight angle.

Ex. 711. Trisect a given right angle.

Ex. 712. Trisect a semicircumference.

Ex. 713. Through a given point, to draw a line cutting off equal lengths on the sides of a given angle.

Ex. 714. Through a given point, to draw a line making a given angle with a given line.

Ex. 715. Through a given point, to draw a line of given length terminating in two given parallel lines.

Ex. 716. Through a given point, to draw a line making equal angles with two given lines.

* **Ex. 717.** To bisect an angle formed by two lines, without producing them to their intersection.

To construct a triangle, having given: (Note 244)

Ex. 718. $a, \angle B, h_a$.

Ex. 719. $a, \angle B, m_c$.

Ex. 720. $a, \angle B, t_c$.

Ex. 721. A, h_a, t_a.

Ex. 722. $a + b + c, \angle B, \angle C$.

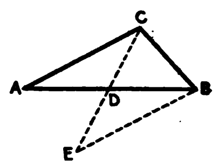
Ex. 723. a, b, R.*

Ex. 724. a, h_a, R.*

Ex. 725. $h_a, h_b, \angle B$.

Ex. 726. a, b, m_c.

(Hint. Produce m_c by its own length.)



* **Ex 727.** $a, m_c, \angle C$.

* **Ex. 728.** m_a, m_b, h_c.

* **Ex. 729.** m_a, h_b, h_c.

* **Ex. 730.** m_a, m_b, m_c.

To construct a square, having given:

Ex. 731. The diagonal.

Ex. 732. The difference between the diagonal and the side.

Ex. 733. The sum of the diagonal and the side.

To construct a rectangle, having given:

Ex. 734. One side and the diagonal.

Ex. 735. One side and the angle formed by the diagonals.

* R = radius of circumscribed circle.

Ex. 736. The perimeter and the diagonal.

To construct a rhombus, having given :

Ex. 737. The two diagonals.

Ex. 738. The perimeter and one diagonal.

Ex. 739. One angle and a diagonal.

Ex. 740. The altitude and the base.

Ex. 741. The altitude and one angle.

To construct a parallelogram, having given :

Ex. 742. Two adjacent sides and one altitude.

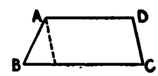
Ex. 743. Two adjacent sides and an angle.

Ex. 744. One side and two diagonals.

Ex. 745. One side, one angle, and one diagonal.

Ex. 746. The diagonals and the angle formed by the diagonals.

269. In the analysis of a problem relating to a trapezoid, draw a line through one vertex, *A*, either parallel to the opposite arm, *DC*, or parallel to a diagonal, *DB*.



To construct a trapezoid, having given :

Ex. 747. The four sides.

Ex. 748. The bases and the base angles.

Ex. 749. The bases, another side, and one base angle.

Ex. 750. The bases and the diagonals.

Ex. 751. One base, the diagonals, and the angle formed by the diagonals.

Ex. 752. To draw a common external tangent to two given circles.

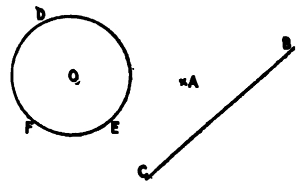
Ex. 753. To draw a common internal tangent to two given circles.

Ex. 754. About a given circle, to circumscribe a triangle, having given the angles.

Ex. 755. Find the locus of the mid-points of the secants that pass through a given point without a circle.

Ex. 756. In a given circle, to inscribe a triangle, having given the angles.

* **Ex. 757.** From a given point in a circumference, to draw a chord that is bisected by a given chord.



Ex. 758. Given a point, A, between a circumference and a straight line Through A, to draw a line terminated by the circumference and the given line, and bisected in A.

Ex. 759. Given two points, A and B, on the same side of a line, CD. To find a point, X, in CD, such that $\angle AXC = \angle BXD$.

[See practical problems 45–52, pp. 293 and 294.]

BOOK III

PROPORTION. SIMILAR POLYGONS

270. Def. A **proportion** is a statement of the equality of two ratios, as $\frac{a}{b}=\frac{c}{d}$ or $a : b = c : d$.

Note. The statements $\frac{a}{b}=\frac{c}{d}$, and $a : b = c : d$, are absolutely identical. Hence, if a hypothesis states $a : b = c : d$, we may in the proof employ this statement in the form $\frac{a}{b}=\frac{c}{d}$ without assigning a special reason. Similarly, if we have to prove $a : b = c : d$, we may prove instead $\frac{a}{b}=\frac{c}{d}$.

271. Def. The first and the fourth terms of a proportion are called the **extremes**, the second and the third, the **means**.

272. Def. The first and the third terms are called the **antecedents**, the second and the fourth the **consequents**.

Thus, in the proportion, $a : b = c : d$, a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

273. Def. When the means of a proportion are equal, *either* mean is said to be the **mean proportional** between the first and the last terms, and the *last* term is said to be the **third proportional** to the first and the second terms.

Thus, in the proportion, $a : b = b : c$, b is the mean proportional between a and c, and c is the third proportional to a and b.

274. Def. The last term is the **fourth proportional** to the first three.

Thus, in the proportion, $a : b = c : d$, d is the fourth proportional to a, b, and c.

275. The two terms of a ratio must be either quantities of the same denomination, or the quantities must be represented by their numerical measures only.

PROPOSITION I. THEOREM

276. *In any proportion, the product of the means is equal to the product of the extremes.*

Given $a : b = c : d$.

To prove $ad = bc$.

Proof. $\frac{a}{b} = \frac{c}{d}$. (Hyp.)

Clearing of fractions, *i.e.* multiplying both members by bd,

$ad = bc$. (Ax. 7.)

Q. E. D.

277. COR. If three terms of a proportion are respectively equal to the three corresponding terms of another proportion, the fourth terms are also equal.

278. NOTE. The product of two quantities, in Geometry, means the product of the numerical measures of the quantities.

Ex. 760. Determine the value of x, if

(*a*) $3 : x = 4 : 8$, (*c*) $x : 7 = 2 : 21$,

(*b*) $112 : 42 = 16 : x$, (*d*) $a : m = x : n$.

Ex. 761. Find the fourth proportional to

(*a*) 1, 2, and 3, (*b*) 2, 1, and 3, (*c*) m, n, and p.

Ex. 762. Find the third proportional to

(*a*) 9 and 12, (*b*) 14 and 21, (*c*) 1 and a.

PROPOSITION II. THEOREM

279. *If the product of two numbers is equal to the product of two other numbers, either two may be made the means, and the other two the extremes, of a proportion.*

Given $mn = pq.$

To prove $m : p = q : n.$

Proof. $mn = pq.$ (Hyp.)

Dividing both members by np

$$\frac{m}{p} = \frac{q}{n}. \qquad \text{(Ax. 8.)}$$

Q. E. D

Ex. 763. If $ab = mn$, find all possible proportions consisting of a, b, m, and n.

Ex. 764. Form two proportions commencing with 3 from the equation $3 \times 10 = 5 \times 6$.

Ex. 765. If $ab = xy$, form two proportions commencing with b.

Ex. 766. Find the ratio of $x : y$, if

(*a*) $6x = 5y$,
(*b*) $9x = 2y$,
(*c*) $6x = y$,
(*d*) $mx = ny$,
(*e*) $(a + b)x = cy$,
(*f*) $mx + nx = py$,
(*g*) $ax + bx = my + ny$,
(*h*) $ax + by = mx + ny$.

Proposition III. Theorem

280. *A mean proportional between two quantities is equal to the square root of their product.*

Given $a : b = b : c.$

To prove $b = \sqrt{ac}.$

Proof. $a : b = c : d.$ (Hyp.)

$\therefore b^2 = ac.$ (276)

Extracting the square root of both members

$$b = \sqrt{ac}.$$

Q. E. D.

Ex. 767. Find the mean proportional to

(*a*) 2 and 18.
(*b*) $\frac{4}{3}$ and $\frac{25}{3}$.
(*c*) $2a$ and $32a$.
(*d*) $m + 1$ and $m - n$.

Proposition IV. Theorem

281. *If four quantities are in proportion, they are in proportion by alternation,* i.e. *the first term is to the third as the second is to the fourth.*

Given $a:b=c:d.$

To prove $a:c=b:d.$

Proof. $\frac{a}{b}=\frac{c}{d}.$ (Hyp.)

$ad=bc.$ (276)

$a:c=b:d.$ (279)

Q. E. D.

Proposition V. Theorem

282. *If four quantities are in proportion, they are in proportion by inversion,* i.e. *the second term is to the first as the fourth is to the third.*

Given $a:b=c:d.$

To prove $b:a=d:c.$

Proof. $a:b=c:d.$ (Hyp.)

$ad=bc.$ (276)

$\therefore b:a=d:c.$ (279)

Q. E. D.

Ex. 768. Transform the proposition, $m:x=p:q$, so that x becomes the fourth term.

Proposition VI. Theorem

283. *If four quantities are in proportion, they are in proportion by composition,* i.e. *the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Given $a:b=c:d.$

To prove $a+b:b=c+d:d.$

Analysis.* The proportion

$$a + b : b = c + d : d$$

is true if $d(a+b) = b(c+d)$. (279)

This is true if $ad + bd = bc + bd$.

This is true if $ad = bc$. (Ax. 3)

This is true if $a : b = c : d$. (276)

But the last equation is true;

$$\therefore a + b : b = c + d : d.$$

The preceding analysis proves the validity of our theorem, but we may obtain a proof in the usual form by reversing the sequence of equations. Hence

Proof. $a : b = c : d$. (Hyp.)

$\therefore ad = bc$. (276)

$\therefore ad + bd = bc + bd$. (Ax. 2)

$\therefore d(a+b) = b(c+d)$ (Sub.)

$\therefore a + b : b = c + d : d$. (279)

NOTE. A much simpler proof, which, however, is more difficult to discover, is the following:

Proof. $\frac{a}{b} = \frac{c}{d}$. (Hyp.)

$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$; (Ax. 2.)

combining, $\frac{a+b}{b} = \frac{c+d}{d}$. Q. E. D.

PROPOSITION VII. THEOREM

284. *If four quantities are in proportion, they are in proportion by division,* i.e. *the difference between the first two terms is to the second term as the difference between the last two terms is to the fourth term.*

* This method may be applied to Props. IV to IX.

Given $a : b = c : d.$

To prove $a - b : b = c - d : d.$

Proof. $\frac{a}{b} = \frac{c}{d}.$

$$\frac{a}{b} - 1 = \frac{c}{d} - 1; \qquad \text{(Ax. 3.)}$$

combining, $\frac{a - b}{b} = \frac{c - d}{d}.$ Q. E. D.

Ex. 769. Make an analysis of Prop. VII and derive a proof from it. (Similar to 283.)

Ex. 770. Transform the following proportions so that only one term contains x.

(a) $2 : 3 = 5 - x : x.$ (d) $4 : 3 = 2 + x : x.$
(b) $6 : 7 = 2 - x : x.$ (e) $7 : 5 = 3 + x : x.$
(c) $a : b = 5 - x : x.$ (f) $a : b = 5 + x : x.$

Ex. 771. If $x + y : y = 7 : 3$, find the ratio of x and y.

Ex. 772. If $x - y : y = 2 : 3$, find the ratio of x and y.

Proposition VIII. Theorem

285. *If four quantities are in proportion, they are in proportion by composition and division,* i.e. *the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given $a : b = c : d.$

To prove $a + b : a - b = c + d : c - d.$

Proof. $a : b = c : d.$ (Hyp.)

$$\frac{a + b}{b} = \frac{c + d}{d}. \qquad (283)$$

$$\frac{a - b}{b} = \frac{c - d}{d}. \qquad (284)$$

Dividing member by member,

$$\frac{a + b}{a - b} = \frac{c + d}{c - d}. \qquad \text{(Ax. 8.)}$$

Q. E. D.

Ex. 773. Make an analysis of Prop. VIII and derive a proof from it. (283)

Ex. 774. Transform the following proportions so that only one term contains x.

$$(a)\ 3:2=5+x:5-x.$$
$$(b)\ 5:3=3+x:3-x.$$
$$(c)\ a:b=1+x:1-x.$$

Ex. 775. If $x+y:x-y=12:5$, find the ratio of x to y.

Ex. 776. If $x+y:\ x-y=a:b$, find the ratio of x to y.

PROPOSITION IX. THEOREM

286. *In a series of equal ratios, the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Given $a:b=c:d=e:f.$

To prove $a+c+e:b+d+f=a:b.$

Analysis. The proportion

$$a+c+e:b+d+f=a:b$$

is true if $ab+bc+be=ab+ad+af.$ (279)

This is true if $bc+be=ad+af.$ (Ax. 3)

But $bc=ad,$ (276)

and $be=af.$ (276)

$$\therefore bc+be=ad+af.$$

Hence $a+c+e:b+d+f=a:b.$

Ex. 777. If $a:b=c:d=e:f=5:7$, find $\dfrac{a+c+e}{b+d+f}$.

Ex. 778. If $\dfrac{x-a-c}{y-b-d}=\dfrac{a}{b}=\dfrac{c}{d}$, find $x:y$.

Ex. 779. If $\dfrac{x}{2}:\dfrac{y}{3}:\dfrac{z}{4}$, find the ratio $x+y+z$ to z.

PROPOSITION X. THEOREM

287. *The products of the corresponding terms of two or more proportions are in proportion.*

Given $a : b = c : d.$

$m : n = p : q.$

To prove $am : bn = cp : dq.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ (Hyp.)

$\frac{m}{n} = \frac{p}{q}.$ (Hyp.)

Multiplying the corresponding members of the equations,

$\frac{am}{bn} = \frac{cp}{dq}.$ (Ax. 7.) Q. E. D.

288. COR. If $\frac{a}{b} = \frac{c}{d},$

then $\frac{ma}{nb} = \frac{mc}{nd}.$

Ex. 780. If $x : y = 1 : 4$, and $x : \frac{1}{y} = 1 : 9$, find x.

Ex. 781. If $\frac{1}{x} : y = 1 : 2$, and $x : y = 1 : 8$, find y.

PROPOSITION XI. THEOREM

289. *If four quantities are in proportion, like powers or like roots of these quantities are in proportion.*

Given $a : b = c : d.$

To prove $a^n : b^n = c^n : d^n$

and $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$

Proof. $\frac{a}{b} = \frac{c}{d}.$

Raising both members to the nth power,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$

Similarly, taking the mth root of each member,

$$\frac{\sqrt[m]{a}}{\sqrt[m]{b}}=\frac{\sqrt[m]{c}}{\sqrt[m]{d}}.$$ Q. E. D.

Ex. 782. If $x^3 : y^3 = 64 : 125$, find $\frac{x}{y}$.

Ex. 783. If $\sqrt{x} : \sqrt{y} = 1 : 2$, find $x : y$.

Ex. 784. If $\sqrt[3]{x} : \sqrt[3]{y} = 1 : 3$, find $x : y$.

Ex. 785. If $\sqrt[3]{x} : 1 = \sqrt[3]{y} : 2$, find $\frac{x}{y}$.

Proposition XII. Theorem

290. *Equimultiples of two quantities are in the same ratio as the quantities.*

Given a and b two quantities.

To prove $ma : mb = a : b$.

[The proof is left to the student.]

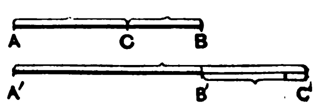
---

291. Def. If in a line AB, or its prolongation, a point C be taken, AC and BC are called **segments of the line.**

292. The segments are **internal** or **external** ones, according as C lies in AB or in the prolongation of AB.

Thus AB is divided internally, $A'B'$ externally.

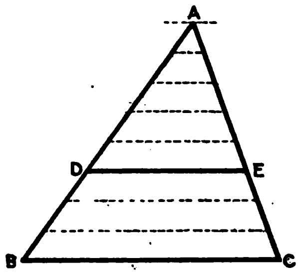
The segments of AB are AC and BC. The segments of $A'B'$ are $A'C'$ and $B'C'$.



PROPORTIONAL LINES

Proposition XIII. Theorem *

293. *A line parallel to one side of a triangle divides the other two sides proportionally.*



Given in $\triangle ABC$, DE parallel to BC.

To prove $AD : DB = AE : EC$.

Proof.† Case I. $\dfrac{AD}{DB}$ is a rational number.

Let $\dfrac{AD}{DB} = \dfrac{5}{3}$, *i.e.* if DB is divided into 3 equal parts, and one of these parts is laid off on AD, then AD contains 5 of these parts.

Through the points of division thus obtained draw parallels to BC.

These lines divide AE into five parts and EC into three parts, all being equal. (152)

Whence $$\frac{AE}{EC} = \frac{5}{3}.$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \quad \text{(Ax. 1.)}$$

Q. E. D.

* For the proof which is based upon the principle of limits, see Appendix.

† See note on p. 123.

CASE II. $\frac{AD}{DB}$ is an irrational number.

Let $\frac{AD}{DB} = \sqrt{3} = 1.732 \cdots$

If $\frac{AD}{DB} = 1.7$, then $\frac{AE}{EC} = 1.7$. (Case I)

If $\frac{AD}{DB} = 1.73$, then $\frac{AE}{EC} = 1.73$, (Case I)

and so forth.

Hence *all* approximate values of $\frac{AD}{DB}$ and $\frac{AE}{EC}$ are respectively equal.

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \quad (223)$$

Q. E. D.

294. REMARK. If the transversal intersects two sides of the triangle, these sides are divided internally; if it meets their prolongations, the sides are divided externally in the same ratio.

295. COR. 1. If a line parallel to one side of a triangle intersects the other two sides, either side is to one of its segments as the other side is to its corresponding segment.

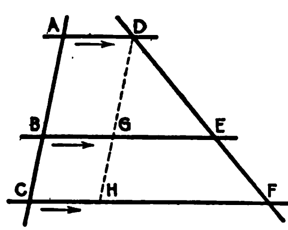
For $AD : DB = AE : EC$.

By composition

$$AD + DB : DB = AE + EC : EC,$$

or $AB : DB = AC : EC$.

296. COR. 2. Three parallel lines cut off proportional segments on any two transversals.



HINT. Draw $DH \parallel AC$, $AB = DG$, $BC = GH$.

Ex. 786. In the diagram for Prop. XIII, if $AD = 4$, $DB = 8$, $AE = 3$, find EC.

Ex. 787. In the same diagram, find DB, if $AD = a$, $AE = b$, and $EC = c$.

Ex. 788. In the same diagram, find AE, if $AB = 12$, $AD = 8$, and $AC = 9$.

Ex. 789. In the same diagram, find AE, if $AB = m$, $AD = n$, and $AC = p$.

Ex. 790. In the same diagram, find AD, if $AD = EC$, $DB = 4$, and $AE = 9$.

Ex. 791. In the same diagram, find AE, if $AE = 2\,DB$, $AD = 10$, and $EC = 20$.

Ex. 792. In the same diagram find EC if $EC = AD$, $AE = m$, and $BD = n$.

Ex. 793. In the same diagram, find EC, if $AB = a$, $AD = b$, and $AC = c$.

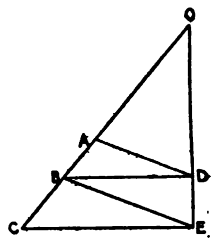
Ex. 794. In the same diagram if $DE \parallel BC$, and $AD : DB = EC : AE$ then $AE = EC$.

Ex. 795. In the same diagram, if $AD = 2(AE)$ and $DB = 6$, find EC.

Ex. 796. If in the diagram for Prop. XIII, $AD = 2, DB = 3$, $AE = 4$, and $EC = 4$, is DE parallel to BC?

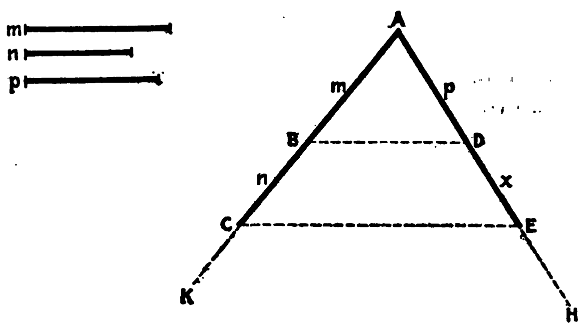
Ex. 797. In the diagram opposite, if $BD \parallel CE$, and $AD \parallel BE$, then

$$OA : OB = OB : OC.$$



Proposition XIV. Problem

297. *To find the fourth proportional to three given lines.*



Given three lines m, n, and p.

Required the fourth proportional to m, n, and p.

Construction. Draw any angle KAH.

On AK, make $AB = m$, $BC = n$; on AH, make $AD = p$.

Draw BD.

Through C, draw a line parallel to BD, meeting AH in E. DE is the required fourth proportional.

[The proof is left to the student.]

Ex. 798. Find the third proportional to two given lines.

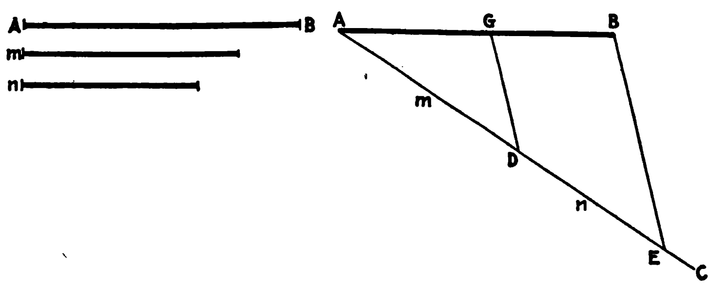
Ex. 799. If a, b, and c are given lines, construct a line x, so that

$$a : b = x : c.$$

Ex. 800. If a, b, and c are given lines, construct a line equal to $\frac{bc}{a}$.

Proposition XV. Problem

298. *To divide a given line into segments proportional to two given lines.*



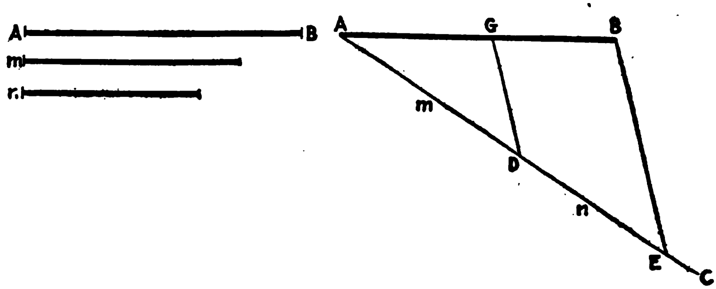
Given lines AB, m, and n.

Required to divide AB into segments proportional to m and n.

Construction. At any angle with AB draw AC.

On AC, cut off $AD = m$, $DE = n$.

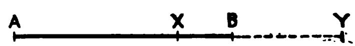
Draw EB.



Through D draw a parallel to BE, intersecting AB in G.
Then AB is divided as required.
[The proof is left to the student.]

299. Def. A line is divided **harmonically** if it is divided internally and externally in the same ratio.

Thus AB is divided harmonically by X and Y if



$$\frac{AX}{BX} = \frac{AY}{BY}.$$

Ex. 801. Given lines AB, m, n, and p. Required to divide AB into three segments AC, CD, and DB, so that

$$AC : m = CD : n = DB : p.$$

Ex. 802. To divide a given line externally into segments proportional to two other given lines.

Ex. 803. To divide a given line AB harmonically in the ratio of two given lines m and n.

Ex. 804. If AB is divided harmonically by two points X and Y, then XY is divided harmonically by A and B (*i.e.* $AX : AY = BX : BY$).

Ex. 805. Construct two lines when their sum and their ratio are given.

Ex. 806. Construct two lines when their difference and their ratio are given.

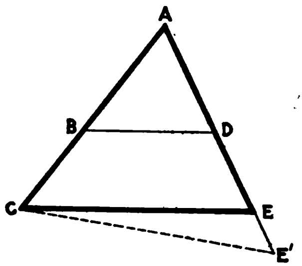
Ex. 807. In a given line, AB, to find a point, C, so that

$$AB : AC = m : n,$$

when m and n are two given lines.

PROPOSITION XVI. THEOREM

300. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given in $\triangle AEC$, $AB : BC = AD : DE$.

To prove DB parallel to EC.

Proof. Through C, draw CE' parallel to BD, meeting AE in E'.

$AB : BC = AD : DE$. (Hyp.)

$AB : BC = AD : DE'$. (293)

$\therefore DE = DE'$. (277)

$\therefore CE$ and CE' coincide.

$\therefore BD \parallel CE$. Q. E. D.

Ex. 808. In the diagram for Prop. XVI, if $AB = 12$, $BC = 16$, $AD = 15$, $DE = 20$, is $BD \parallel CE$?

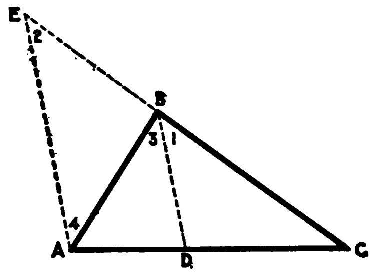
Ex. 809. Demonstrate that there is only one point that divides a given line internally in a given ratio.

Ex. 810. The four points M, N, P, Q lie respectively on the sides AB, BC, CD, DA of quadrilateral $ABCD$, $AM : MB = AQ : QD$, and $BN : NC = DP : PC$. Prove that MQ is parallel to NP.

Ex. 811. On the side AB of $\triangle ABC$ the point D and E be taken so that $AD = BE$. Through D a parallel is drawn to BC, that meets AC in F, and through E a line is drawn parallel to AC, that meets BC in G. Prove that $FG \parallel AB$.

Proposition XVII. Theorem

301. *The angle bisector of a triangle divides the opposite side into segments which are proportional to the other two sides.*



Given in $\triangle ABC$, BD bisecting $\angle ABC$.

To prove $AB : BC = AD : DC$.

Proof. Draw $AE \parallel DB$, to meet CB, produced, in E.

	$\angle 1 = \angle 2.$	(106)
	$\angle 3 = \angle 4.$	(104)
But	$\angle 1 = \angle 3.$	(Hyp.)
	$\therefore \angle 2 = \angle 4.$	(Ax. 1.)
	$\therefore AB = BE.$	(119)
But	$EB : BC = AD : DC.$	(293)
	$\therefore AB : BC = AD : DC.$	(Sub.)

Q. E. D.

Ex. 812. In the diagram for Prop. XVII, find DC, if $AB = 3$, $BC = 4$, and $AD = 2$.

Ex. 813. In the same diagram, find AD, if $AB = m$, $BC = n$, and $DC = p$.

Ex. 814. In the same diagram, find DC, if

(a) $AB = 4$, $BC = 5$, $AC = 6$.
(b) $AB = 18$, $BC = 9$, $AC = 21$.
(c) $AB = 21$, $BC = 14$, $AC = 25$.

Ex. 815. In the same diagram, find AD and DC, if $BC = a$, $CA = b$, and $AB = c$.

Ex. 816. In the same diagram, find AB, if $AB = DC$, $AD = 4$, and $BC = 16$.

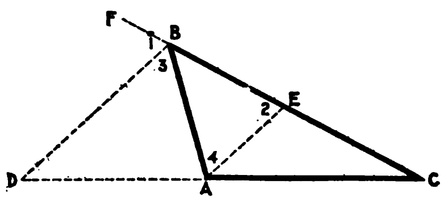
Ex. 817. In the same diagram, find AB, if $AB = DC$, $AD = m$, and $BC = n$.

Ex. 818. To divide one side of a triangle into segments proportional to the other two sides.

Ex. 819. State and prove the converse of Prop. XVII.

Proposition XVIII. Theorem

302. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the other two sides.*



Given $\triangle ABC$, BD bisecting the exterior $\angle ABF$.

To prove $AB : BC = AD : DC$.

Hint. This proof is almost literally the same as the proof of Prop. XVII.

Ex. 820. If in the diagram for Prop. XVIII $BA = 3$, $BC = 4$, $AC = 5$, find DA.

Ex. 821. In the same diagram find DA, if $DA = BC$, $BA = 4$, and $DC = 9$.

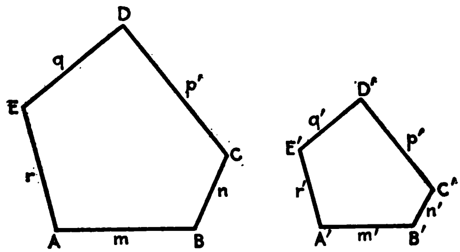
Ex. 822. In the same diagram find DC, if $BC = a$, $AC = b$, and $AB = c$.

Ex. 823. State and prove the converse of Prop. XVIII.

Ex. 824. The bisectors of an interior and an adjacent exterior angle of a triangle divide the opposite side harmonically.

SIMILAR POLYGONS

303. DEF. Two polygons are **similar** if their homologous angles are equal and their homologous sides proportional.



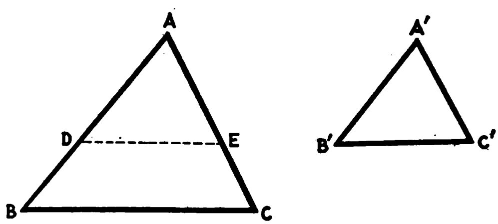
Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar, if

(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = C'$, etc.

(2) $\frac{m}{m'} = \frac{n}{n'} = \frac{p}{p'}$, etc.

PROPOSITION XIX. THEOREM

304. *Two triangles are similar if the three angles of one are respectively equal to the three angles of the other.*



Given in $\triangle$ ABC and $A'B'C'$,

$$\angle A = \angle A',\ \angle B = \angle B',\text{ and }\angle C = \angle C'.$$

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$. Then B' will fall on some point D and C' on a point E.

Since $\angle ADE = \angle B$, (Hyp.)

$DE \parallel BC$. (95)

$\therefore \frac{AB}{AD} = \frac{AC}{AE}$. (295)

Or $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. (Sub.)

In a similar manner it can be shown that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$. (Ax. 1.)

$\therefore \triangle ABC \sim \triangle A'B'C'$. (303)

Q. E. D.

305. Cor. 1. **Two triangles are similar if two angles of the one are respectively equal to two angles of the other.**

306. Cor. 2. **Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.**

Ex. 825. The sides of a polygon are 3, 4, 5, 6, and 7. Find the sides of a similar polygon if the side corresponding with 3 equals 15.

Ex. 826. The sides of a triangle are a, b, and c. Find the sides of a similar triangle if the side corresponding with a is equal to m.

Ex. 827. If two chords AB and CD intersect in E, the triangle AEC is similar to the triangle BED.

Ex. 828. If from a point A without a circle, two secants are drawn to meet the circumference in B and C, and D and E respectively, the triangle ABE is similar to the triangle ACD.

Ex. 829. If the altitudes AD and BE of the triangle ABC intersect in F, the triangle AFE is similar to the triangle BFD.

Ex. 830. If AD, the angle bisector of the inscribed triangle ABC, be produced to meet the circumference in E, the triangle ABD is similar to the triangle AEC.

Ex. 831. Two isosceles triangles are similar if an angle of the one is equal to the homologous angle of the other.

307. METHOD XVII. **Similar triangles are the usual means of proving that lines are proportional.** To prove, therefore, that four lines are proportional:

(1) *Select two triangles so that each contains two of the given lines.* (It is advisable to mark the lines as indicated in 256 and 257.)

(2) *Prove the similarity of the two triangles.* (If triangles are not similar, select another pair.)

(3) *Derive the proportion.*

(4) (*Apply alternation and inversion,* if necessary.)

Ex. 832. If in the triangle ABC the altitudes AD and BE be drawn, prove that $AC : BC = DC : EC$.

Ex. 833. In the same diagram, if AD and BE meet in F, prove $BF : FA = DF : FE$.

Ex. 834. In the same diagram, $AE : AD = FE : DC$.

Ex. 835. If from the vertex A of an inscribed triangle ABC the altitude AD and the diameter AF be drawn, then $AB : AD = AF : AC$.

Ex. 836. In the same diagram, $BD : FC = AD : AC$.

Ex. 837. If from a point without a circle a tangent and a secant be drawn, the tangent is the mean proportional between the secant and its exterior segment.

Ex. 838. If a diameter AB be produced to C, at C a perpendicular be erected, and through B a line be drawn to meet the circumference and the perpendicular in D and E respectively, then $AB : BE = DB : BC$.

Ex. 389. The diagonals of a trapezoid divide each other proportionally.

Ex. 840. If in a right triangle ABC the altitude AD be drawn upon the hypotenuse, $AD : AB = AC : BC$.

Ex. 841. In the same diagram, $AD : AB = DC : AC$.

Ex. 842. If the angle bisector CD of an inscribed triangle ABC be produced to meet the circle in E, $EB : EC = DB : CB$.

Ex. 843. In the same diagram, $AD : EB = AC : CE$.

Ex. 844. In similar triangles homologous angle bisectors have the same ratio as any two homologous sides.

Ex. 845. In similar triangles homologous altitudes have the same ratio as any two homologous sides.

308. METHOD XVIII. **To prove that the product of two lines equals the product of two other lines,** *use the method of* (307) *and take the products of the means and extremes of the resulting proportion.*

Ex. 846. If two chords intersect within a circumference, the product of the segments of one is equal to the product of the segments of the other.

Ex. 847. If from any point E in the chord AB the perpendicular EC be drawn upon the diameter AD, then

$$AC \times AD = AB \times AE.$$

Ex. 848. The product of two arms of a right triangle is equal to the product of the hypotenuse and the altitude upon the hypotenuse.

Ex. 849. If in triangle ABC, a parallel to AB meet BC and CA in E and D respectively, then

$$AC \times DE = DC \times AB.$$

Ex. 850. The product of any altitude of a triangle and its corresponding side is equal to the product of any other altitude and its corresponding side.

Ex. 851. The three sides of triangles are 14, 15, 13, and the altitude upon 14 equals 12. Find the other altitudes. (Compare preceding exercise.)

Ex. 852. If in triangle ABC the altitudes AD and BE meet in F, then $BF \times BE = BC \times BD$.

Ex. 853. If in the triangle ABC the altitudes AD and BE meet in F, then

$$BD \times DC = DF \times AD.$$

Ex. 854. If AB is a diameter, BD the tangent at B, and DA meets the circumference in E, then

$$\overline{AB}^2 = AE \times AD$$

Ex. 855. In the same diagram,

$$\overline{BE}^2 = AE \times ED.$$

Ex. 856. If the sides AB and DC of inscribed quadrilateral $ABCD$ are produced until they meet in E, and $\angle DBA = \angle CBE$, then $AD \times BE = CE \times BD$.

Ex. 857. In the same diagram, $EC \times AD = BC \times EA$.

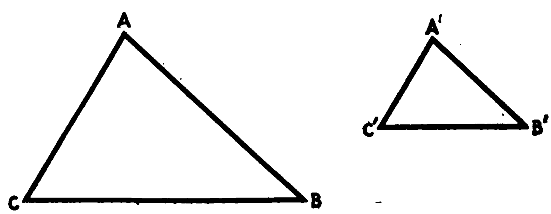
Ex. 858. If in $\triangle ABC$ the altitudes AD and BE are drawn and $BE = 6$, $EC = 3$, $DC = 2$, find AD.

***Ex. 859.** If AE touches a circle at A, AB is a chord, BC is a perpendicular from B upon AE, $AC = 4$, and $AB = 6$, find the diameter of the circle.

Ex. 860. Upon a given line $A'B'$ to construct a triangle similar to a given triangle ABC, so that $A'B'$ and AB become homologous sides.

Proposition XX. Theorem

309. *Two triangles are similar if an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*



Given in $\triangle s$ ABC and $A'B'C'$,

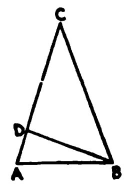
$$\angle A = \angle A', \text{ and } AB : A'B' = AC : A'C'.$$

To prove $\triangle ABC \sim \triangle A'B'C'$.

Hint. Place $\triangle A'B'C'$ upon $\triangle ABC$, so that A' coincides with A, and prove that $C'B'$ will become $\parallel CB$.

Ex. 861. Two isosceles triangles are similar if their vertex angles are equal.

Ex. 862. In similar triangles, homologous medians have the same ratio as any two homologous sides.



Ex. 863. If in the annexed diagram, AB is the mean proportional between AD and AC, $\triangle ABD \sim \triangle ACB$.

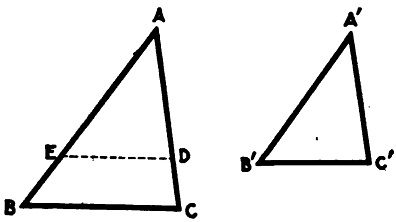
Ex. 864. In the same diagram, if $AC = BC$, and $\overline{AB}^2 = AD \times AC$, prove that $AB = BD$.

Ex. 865. Two triangles, ABC and $A'B'C'$, are similar if altitude AD : altitude $A'D' = BC : B'C'$, and $\angle B = \angle B'$.

Ex. 866. In similar triangles, the radii of the circumscribed circles have the same ratio as any two homologous sides.

Proposition XXI. Theorem

310. *Two triangles are similar if their homologous sides are proportional.*



Given in $\triangle$ ABC and $A'B'C'$, $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. On AC and AB respectively, lay off $AD = A'C'$ and $AE = A'B'$, and draw DE.

Then $\triangle ADE \sim \triangle ACB$ (309)

$\therefore AB : AE = BC : ED,$ (303)

but $AB : A'B' = BC : B'C'.$ (Hyp.)

$\therefore$ since $AE = A'B'$,

$ED = B'C'.$ (277)

$\therefore \triangle A'B'C' \cong \triangle ADE,$ (s.s.s. = s.s.s.)

But $\triangle ADE$ has been proven similar to $\triangle ABC$.

$\therefore \triangle A'B'C' \sim \triangle ABC$ (Sub.)

Q.E.D.

Ex. 867. The lines which join the mid-points of the sides of $\triangle ABC$ form a triangle similar to ABC.

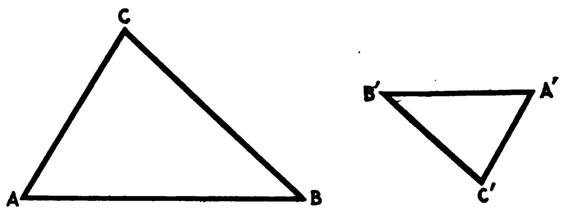
Ex. 868. Two triangles are similar if two sides and the median to one of these sides of one are proportional to the homologous parts of the other triangle.

* **Ex. 869.** Two triangles are similar if two sides and the radius of the circumscribed circle of the one are proportional to the homologous parts of the other.

* **Ex. 870.** Two right triangles are similar if the hypotenuse and an arm of one triangle are proportional to the hypotenuse and an arm of the other.

Proposition XXII. Theorem

311. *Two triangles are similar if the sides of one are respectively parallel to the sides of the other.*



Given in $\triangle$s ABC and $A'B'C'$, $AB \parallel A'B'$, $AC \parallel A'C'$, and $BC \parallel B'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. $\angle$s A and A' are equal or supplementary. (109)

In like manner, $\angle$s B and B', and C and C' are either equal or supplementary.

Hence, one of the following possibilities must be true:

1. The three homologous angles are supplementary,

i.e. $\angle A + \angle A' = 2$ rt. $\angle$s, $\angle B + B' = 2$ rt. $\angle$s, $\angle C + \angle C' = 2$ rt. $\angle$s.

2. Two angles are supplementary, one is equal to the homologous one,

e.g. $\angle A + \angle A' = 2$ rt. $\angle$s, $\angle B + \angle B' = 2$ rt. $\angle$s, $\angle C = \angle C'$.

3. Two angles are equal, one is supplementary to the homologous ones,

$$e.g.\ \angle A + \angle A' = 2 \text{ rt. } \angle s,\ \angle B = \angle B',\ \angle C = \angle C'.$$

4. The angles are respectively equal,

$$i.e.\ \angle A = \angle A',\ \angle B = \angle B',\ \angle C = \angle C'.$$

The first two statements cannot be true, for the sum of the angles of the two triangles would exceed four right angles.

Therefore, two angles of one triangle are equal to two angles of the other.

$$\therefore \triangle ABC \sim \triangle A'B'C'. \qquad \text{Q.E.D.}$$

Ex. 871. Two triangles are similar if the sides of one are respectively perpendicular to the sides of the other.

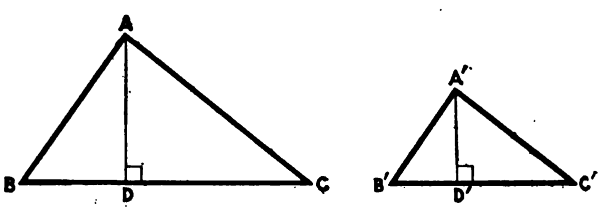
Ex. 872. If a point O within triangle ABC is joined with A, B, and C respectively, and A', B', C', are respectively the mid-points of OA, OB, and OC, then $\triangle A'B'C' \sim \triangle ABC$.

Ex. 873. If a point O without triangle ABC is joined to A, B, and C, and upon the three lines OA, OB, and OC, three points A', B', C' are respectively taken, so that $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC}$, then $\triangle A'B'C' \sim \triangle ABC$.

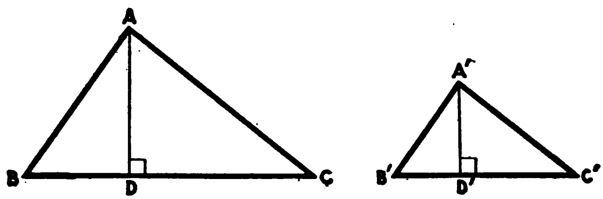
[See practical problems 54–60, pp. 294 and 295.]

Proposition XXIII. Theorem

312. *If two triangles are similar, homologous altitudes have the same ratio as any two homologous sides.*



Given AD and $A'D'$, the homologous altitudes of the similar triangles ABC and $A'B'C'$.



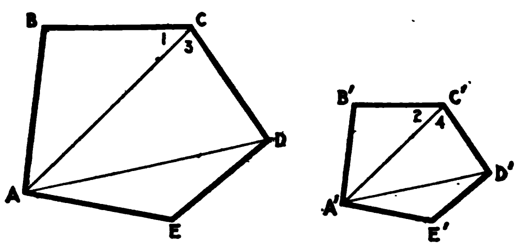
To prove $$\frac{AD}{A'D'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{C'A'}.$$

Hint. What is the means of proving that four lines are proportional?

Ex. 874. The base of a triangle is 2 ft. and the altitude 9 in. If the homologous base of a similar triangle is 6 in., find the homologous altitude.

Proposition XXIV. Theorem

313. *Two similar polygons may be divided into the same number of similar triangles similar each to each and similarly placed.*



Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

To prove $\triangle ABC \sim \triangle A'B'C'$,

$\triangle ACD \sim \triangle A'C'D'$, etc.

Proof. Since the polygons are similar by hyp.,

$AB : A'B' = BC : B'C'$ and $\angle B = \angle B'$. (303)

Hence $\triangle ABC \sim \triangle A'B'C'$. (309)

But $\angle BCD = \angle B'C'D'$, (Hyp.)

and $\angle 1 = \angle 2.$ (303)

$\therefore \angle 3 = \angle 4.$ (Ax. 3.)

Since the polygons are similar, and $\triangle ABC \sim \triangle A'B'C'$,

$$\frac{BC}{B'C'} = \frac{CD}{C'D'}, \quad (303)$$

and

$$\frac{BC}{B'C'} = \frac{AC}{A'C'}. \quad (303)$$

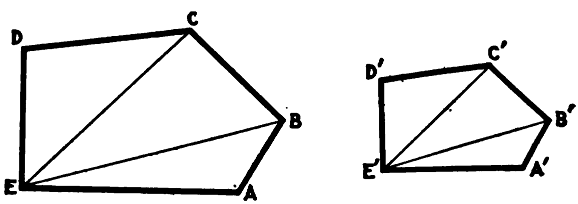
$$\therefore \frac{CD}{C'D'} = \frac{AC}{A'C'}. \quad (\text{Ax. } 1.)$$

$$\therefore \triangle ACD \sim \triangle A'C'D'. \quad (309)$$

In like manner, $\triangle$s ADE and $A'D'E'$ are similar. Q. E. D.

Proposition XXV. Theorem

314. *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*



Given in the polygons $ABCDE$ and $A'B'C'D'E'$,

$\triangle ABE \sim \triangle A'B'E'$,

$\triangle BCE \sim \triangle B'C'E'$,

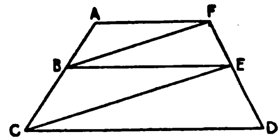
$\triangle CDE \sim \triangle C'D'E'$, etc.

To prove polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

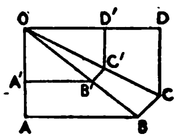
Hint. The polygons are mutually equiangular by Axiom 2.

The ratio of any pair of homologous sides is equal to the ratio of the next pair, for either ratio is equal to the ratio of the included homologous diagonals.

Ex. 875. If $AF \parallel BE \parallel CD$ and $BF \parallel CE$, prove that $ABEF$ is similar to $BCDE$. (See diagram.)



Ex. 876. If $AB \parallel A'B'$, $BC \parallel B'C'$, and $CD \parallel C'D'$, then $OABCD$ is similar to $OA'B'C'D'$.



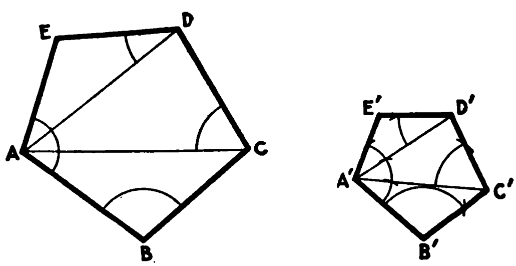
Ex. 877. Polygon $ABCDE$ is similar to polygon $A'B'C'D'E'$ if the sides of the first are respectively parallel to the sides of the second and if $AC \parallel A'C'$, $AD \parallel A'D'$.

Ex. 878. If polygon $ABCDEF$ is similar to polygon $A'B'C'D'E'F'$, then $ABCDE$ is similar to $A'B'C'D'E'$.

[See practical problems, 61–63, p. 295.]

Proposition XXVI. Problem

315. *To construct a polygon similar to a given polygon upon a line homologous to a side of the given polygon.*



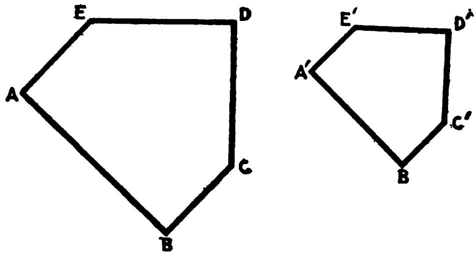
Given polygon $ABCDE$, and line $A'B'$.

Hint. Draw AD and AC and make corresponding $\angle s$ equal.

Ex. 879. To construct a quadrilateral $ABCD$ similar to a given quadrilateral at $A'B'C'D'$ and having the diagonal equal to a given line.

PROPOSITION XXVII. THEOREM

316. *The perimeters of two similar polygons are to each other as any two homologous sides.*



Given P and P', the perimeters of the similar polygons $ABCDE$ and $A'B'C'D'E'$ respectively.

To prove $P : P' = AB : A'B'$.

HINT. $AB : A'B' = BC : B'C' = CD : C'D' = DE : D'E' = EA : E'A'$. Apply (286).

Ex. 880. The sides of a polygon are 4, 5, 6, 7, and 8 respectively. Find the perimeter of a similar polygon, if the side corresponding to 5 is 7.

Ex. 881. The perimeters of two similar polygons are 20 and 25 in. respectively. If a side of the first polygon is 4 in., find the homologous side of the second polygon.

Ex. 882. The perimeters of two similar polygons are to each other as any two homologous diagonals.

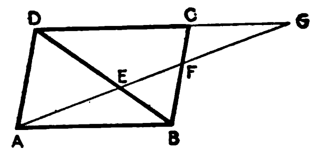
Ex. 883. The perimeters of two similar triangles are to each other as any two homologous altitudes.

Ex. 884. In the diagram for Prop. XVII find the perimeter of $ABCDE$, if the perimeter of $A'B'C'D'E'$ equals 20 inches, $A'B' = 4$ inches, $B'C' = 3$ inches, $AC = 10$ inches, $\angle B = 90°$, and $ABCDE \sim A'B'C'D'E'$.

317. METHOD XIX. **To prove the proportionality of four lines which do not form similar triangles, find a third ratio equal to each of the given ones.**

Thus, in the annexed figure, if $ABCD$ is a parallelogram and AG is a straight line, then

$$\frac{EF}{EA}=\frac{EA}{EG}.$$



Obviously no two triangles exist, each of which contains two of the four lines. Hence we have to find a third ratio equal to each of the given ones. This ratio is $\frac{EB}{ED}$. Our problem is therefore split into two, viz.:

(*a*) To prove $$\frac{EF}{EA}=\frac{EB}{ED},$$

and

(*b*) To prove $$\frac{EA}{EG}=\frac{EB}{ED}.$$

Each of these proportions is easily proved by means of the fundamental method (XVII).

Ex. 885. In similar triangles the radii of the inscribed circles have the same ratio as any two homologous sides.

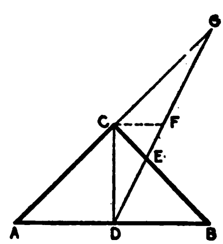
Ex. 886. If triangle ABC is similar to triangle $A'B'C'$, and AD and $A'D'$ are angle bisectors, prove that

$$AD : A'D' = BC : B'C'.$$

Ex. 887. If in the similar triangles ABC and $A'B'C'$, the points D and D' are taken respectively in BC and $B'C'$ so that $\angle BAD=\angle B'A'D'$, then $BD : B'D' = BC : B'C'$.

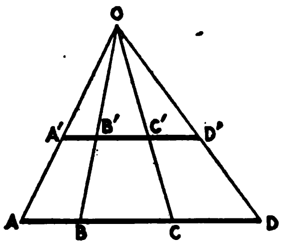
Ex. 888. If D is the mid-point of AB, $CF \parallel AB$, and DG is a straight line, prove that

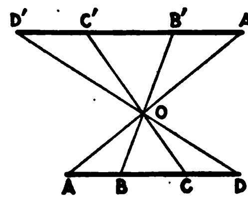
$$\frac{DE}{FE}=\frac{DG}{FG}.$$ (Annexed diagram.)



PROPOSITION XXVIII. THEOREM

318. *If two parallel lines are cut by three or more transversals passing through a common point, the corresponding segments are proportional.*





Given the transversals OA, OB, OC, and OD intersecting the parallel lines AD and $A'D'$ in A, B, C, D, and A', B', C', D' respectively.

To prove $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}.$$

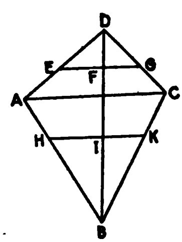
HINT. Which method for demonstrating the proportionality of the first four lines must be applied? Why?

$$\left(\frac{OB}{OB'} \text{ is the third ratio.}\right)$$

Ex. 889. In the diagram for Prop. XXVIII, if $AB = B'C'$, $A'B' = 4$, and $BC = 9$, what is the value of AB?

Ex. 890. In quadrilateral $ABCD$, if $EG \parallel AC \parallel HK$, prove that

$$\frac{EF}{FG}=\frac{HI}{IK}.$$



* **Ex. 891.** In the same diagram, if

$$EG \parallel AC \parallel HK$$

and $\quad EH \parallel DB,$

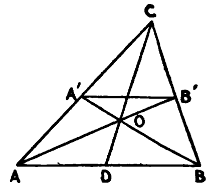
prove that $\quad FG = IK,$

and $\quad EH \parallel GK.$

* **Ex. 892.** If the non-parallel sides of trapezoid $ABB'A'$ meet in C, and a line drawn from C through the intersection of the diagonals O meets AB in D, prove that

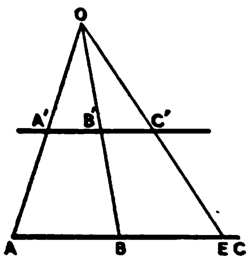
$$AD = DB.$$

(HINT. Prove $AD : DB = DB : AD$.)



PROPOSITION XXIX. THEOREM

319. *If three or more non-parallel transversals intercept proportional segments on two parallel lines, they intersect in a common point.**



Given AA', BB', and CC' intersecting the parallels AC and $A'C'$, so that $AB : A'B' = BC : B'C'$.

To prove AA', BB', and CC' intersect in a common point.

Proof. Let AA' and BB' intersect in O.

Draw OC', and suppose that its prolongation meets AC in E.

Then $AB : A'B' = BC : B'C'$. (Hyp.)

$AB : A'B' = BE : B'C'$. (318)

$\therefore BE = BC$, (277)

or E coincides with C.

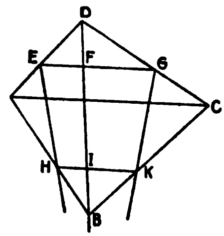
$\therefore C$, C', and O lie in a straight line,

or CC', produced, passes through O. Q. E. D.

* See footnote, p. 45.

Ex. 893. Given two lines, AB and CD, and point E. Without producing AB and CD to their point of intersection, to draw a line XY through E, so that AB, CD, and XY would meet in a common point.

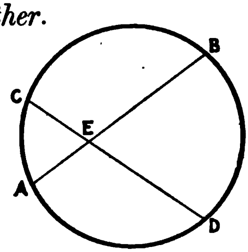
Ex. 894. If in quadrilateral $ABCD$, $EG \parallel AC \parallel HK$, then EH, FI, and GK produced meet in a common point or are parallel.



[See applied problem 64, p. 297.]

Proposition XXX. Theorem

320. *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Given in $\odot O$, the chords AB and CD intersecting in E.

To prove $AE \times EB = CE \times ED$.

Hint. What is the means of proving that the product of two lines is equal to the product of two other lines?

Ex. 895. In the diagram for Prop. XXX, if $AE = 3$, $EB = 4$, $ED = 6$, find CE.

Ex. 896. In the same diagram, if $AE = a$, $EB = b$, and $ED = c$, find CE.

Ex. 897. In the same diagram, if $AE = 4$, $EB = 9$, and $CE = ED$, find CE.

Ex. 898. If the prolongations of two chords meet without a circle, is Prop. XXX correct for the external segments of the chords?

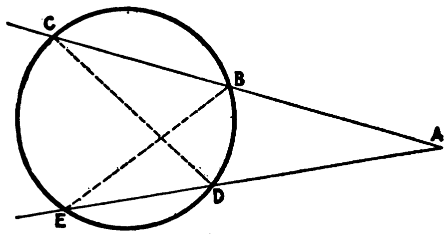
Ex. 899. If two lines AB and CD intersect in E so that

$$AE \times EB = CE \times ED,$$

then A, B, C, and D are concyclic. (See Ex. 581.)

PROPOSITION XXXI. THEOREM

321. *If from a point without a circle, two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*



Given two secants AC and AE cutting a circle in B, C, and D, E respectively.

To prove $AC \times AB = AE \times AD$.

[The proof is left to the student.]

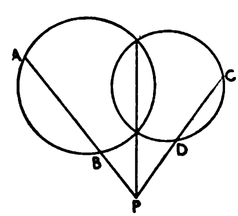
Ex. 900. If, in the diagram for Prop. XXXI, $CB = 16$, $BA = 2$, $DA = 4$, find DE.

Ex. 901. In the same diagram, if $CA : DA = 2 : 1$, prove that $EA = 2\,(BA)$.

Ex. 902. If in the same diagram $CB = a$, $BA = b$, $AE = c$, find DA.

Ex. 903. In the same diagram, if $AB = AD$, then $BCED$ is an isosceles trapazoid.

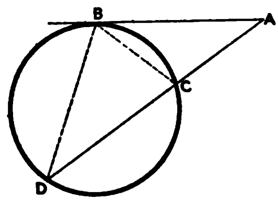
Ex. 904. In the same diagram, if $CB = 5$ inches, $BA = 3$ inches, and the distance of A from the center of the circle equals 7 inches, find the radius of the circle.



Ex. 905. If two circles intersect and from a point P in the prolongation of the common chord the secants PA and PC are drawn, then $PA \times PB = PC \times PD$.

PROPOSITION XXXII. THEOREM

322. *If from a point without a circle, a tangent and a secant are drawn, the tangent is the mean proportional between the secant and its external segment.*



Given the tangent AB touching the $\odot BDC$ in B, and the secant AD cutting the circle in C and D.

To prove $AD : AB = AB : AC.$

[The proof is left to the student.]

Ex. 906. If in the diagram for Prop. XXXII, $DC = 5$, $AC = 4$, find AB.

Ex. 907. In the same diagram, if $AB = a$, and $AC = b$, find AD.

Ex. 908. In the same diagram, if $AB = 8$ inches, and the distance of A from the center of the circle equals 17 inches, find the radius of the circle.

Ex. 909. Construct the mean proportional between two given lines by means of Prop. XXXII.

Ex. 910. Tangents to two intersecting circles drawn from any point in the common chord, produced, are equal.

Ex. 911. If two circles are tangent externally and from any point in the common internal tangent secants are drawn to the two circles, the products of the secants and their external segments will be equal.

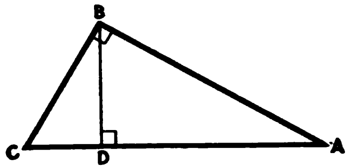
* **Ex. 912.** To construct a circle passing through two given points and touching a given line.

Ex. 913. If the diameter of the earth is equal to 8000 mi., how far can you see from a lighthouse 100 ft. high?

[For additional practical applications, see problems 65–71, pp. 297 and 298.]

Proposition XXXIII. Theorem

323. *In a right triangle, the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse, and either arm is the mean proportional between the hypotenuse and the adjacent segment.*



Given in the rt. $\triangle ABC$, BD the altitude upon the hypotenuse AC.

To prove (1) $AD : DB = DB : DC$.

(2) $AD : AB = AB : AC$.

Hint. $\triangle ABD \sim \triangle ABC$. (Why ?)

$\triangle CBD \sim \triangle ABC$. (Why ?)

$\therefore \triangle ABD \sim \triangle CBD$.

$\therefore AD : DB = DB : DC$,

and $AD : AB = AB : AC$. Q. E. D.

324. Cor. The perpendicular from any point in the circumference upon the diameter is the mean proportional between the segments of the diameter; and the chord joining the point to either extremity is the mean proportional between the diameter and the adjacent segment.

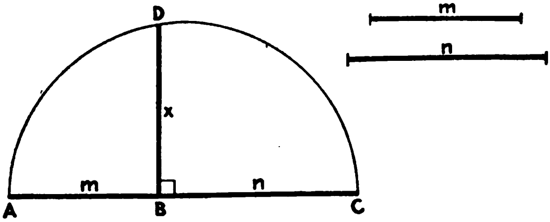
Ex. 914. In the diagram for Prop. XXXIII, if $AD = 4$ and $AC = 9$, what is the value of AB?

Ex. 915. In the same diagram, if $AB = 15$, $AC = 25$, find AD and BD.

Ex. 916. In the same diagram, if $BD = 24$, $AD = 18$, find AC and BC.

PROPOSITION XXXIV. PROBLEM

325. *To construct the mean proportional between two given lines.*



Given lines m and n.

Required the mean proportional between m and n.

Construction I. Draw $AB = m$, and produce AB to C so that $BC = n$.

On AC as a diameter, describe a semicircle.

At B, erect a perpendicular upon AC, meeting the circle in D.

BD is the required mean proportional.

[The proof is left to the student.]

Construction II. Draw $AB = n$.

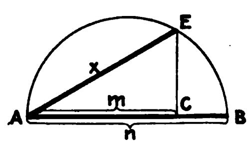
On AB as a diameter, describe a semicircle.

On AB lay off $AC = m$.

Draw $CE \perp AB$. Join A and E.

AE is the required mean proportional.

[The proof is left to the student.]



Ex. 917. If a and b are given lines, construct $\sqrt{ab}$.

Ex. 918. Construct $\sqrt{6\,ab}$ if a and b are two given lines.

Ex. 919. Construct a line equal to $a\sqrt{2}$ if a is a given line.

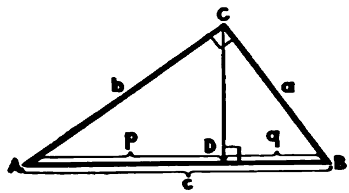
(HINT. $a\sqrt{2} = \sqrt{(2\,a)\cdot a}$).

Ex. 920. Construct a line equal to $a\sqrt{5}$ if a is a given line.

Ex. 921. Construct a line equal to $a\sqrt{\frac{5}{7}}$ if a is a given line.

PROPOSITION XXXV. THEOREM

326. *The sum of the squares of the arms of a right triangle is equal to the square of the hypotenuse.*



Given ABC a rt. △, having its rt. ∠ at C.

To prove $a^2 + b^2 = c^2$.

Proof. Draw $CD \perp AB$, and denote AD by p, and DB by q.

$$p : b = b : c. \qquad (223)$$

$$\therefore b^2 = p \cdot c. \qquad (276)$$

Similarly $a^2 = q \cdot c$.

By adding $a^2 + b^2 = c(p + q)$,

or $a^2 + b^2 = c^2$. Q. E. D.

327. COR. The square of either arm of a right triangle is equal to the square of the hypotenuse, diminished by the square of the other arm.

Ex. 922. Find the hypotenuse of a right triangle whose arms are respectively

(a) 1 ft. and 5 in. (b) m and n.

Ex. 923. The hypotenuse of a right triangle is 25, one arm equals 20; find the other arm.

Ex. 924. Find the altitude of an equilateral triangle whose side is equal to 8 in.

Ex. 925. Find the altitude of an equilateral triangle whose side is equal to a.

Ex. 926. Find the altitude of an isosceles triangle whose base equals 8 and whose arm equals 5.

Ex. 927. If the hypotenuse of an isosceles right triangle equals 8 in., what is the length of an arm?

Ex. 928. The radii of 2 circles are respectively 6 in. and 21 in., and the distance between their centers is 25 in.; find the length of the common external tangent.

Ex. 929. Find the side of an equilateral triangle whose altitude equals 10.

Ex. 930. The squares of the two arms of a right triangle have the same ratio as the adjacent segments of the hypotenuse.

Ex. 931. If AD is an altitude of a triangle ABC,

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Ex. 932. If the diagonals of a quadrilateral are perpendicular to each other, then the sum of the squares of two opposite sides is equal to the sum of the squares of the other two.

Ex. 933. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.

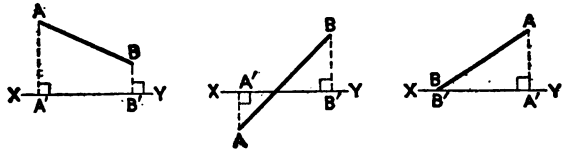
HINT. Draw a rt. △ whose arms are respectively equal to the arms of the given △, and prove the equality of the two ▲.

* **Ex. 934.** If the square of one side of a triangle is greater than the sum of the squares of the other two, the triangle is obtuse.

HINT. Compare the △ with a rt. △ which has the same arms as the given one.

328. DEF. **The projection of a point upon a line** is the foot of the perpendicular from the point to the line.

329. DEF. **The projection of one line upon another** is the segment between the projections of the extremities of the first line upon the second.

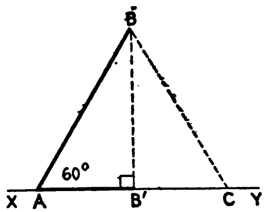


Thus, if $AA' \perp XY$ and $BB' \perp XY$, A' is the projection of A upon XY, and $A'B'$ is the projection of AB upon XY.

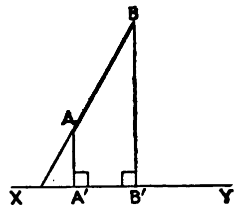
Ex. 935. In acute triangle ABC, draw the projection of AB upon AC, of AB upon BC, of AC upon AB.

Ex. 936. If $AB \parallel XY$, prove that the projection of AB upon XY equals AB.

Ex. 937. If the side of an equilateral triangle equals 10 in., what is the length of the projection of one side upon another?



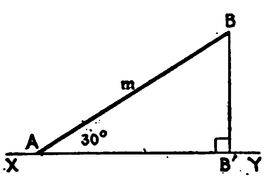
Ex. 938



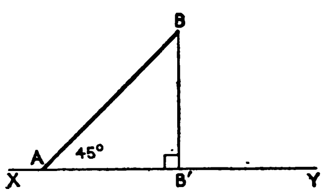
Ex. 939

Ex. 938. If the lines AB and XY include an angle of 60°, the projection of AB upon XY equals one half AB (167).

Ex. 939. Prove that the projection $A'B'$ of AB upon XY equals one half AB, if the *prolongation* of BA forms an angle of 60° with XY.



Ex. 940



Ex. 941

Ex. 940. If the lines AB and XY include an angle of 30°, and $AB = m$, prove that the projection of AB upon XY equals $\frac{m}{2}\sqrt{3}$.

HINT. $BB' = \frac{m}{2}$. (Ex. 937.)

Ex. 941. If the lines AB and XY include an angle of 45°, and $AB = m$, prove that the projection of AB upon $XY = \frac{m}{2}\sqrt{2}$.

HINT. If $AB' = x$, then $BB' = x$. $\therefore x^2 + x^2 = m^2$.

Ex. 942. Prove the last two exercises, if the *prolongation* of AB forms an angle of 30°; an angle of 45°. (Diagram similar to Ex. 939.)

Ex. 943. Find the projection of AB upon XY if $AB = m$, and the angle included by AB and XY equals 120°.

Ex. 944. Find the projection of AB upon XY if $AB = m$, and the angle included by AB and XY equals 135°.

Ex. 945. Find the projection of AB upon XY if $AB = m$, and the angle included by AB and XY equals 150°.

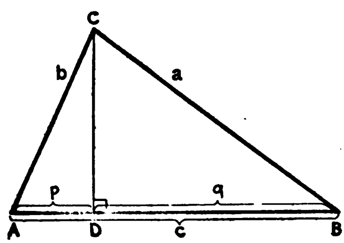
Ex. 946. If in triangle ABC, $AB = 8$, $AC = 10$, and $\angle A = 60°$, find the projection of AB upon AC, of BC upon AC.

Ex. 947. If in triangle ABC, $AB = 10$, $AC = 12$, and $\angle A = 45°$, find the projection of AB upon AC.

Ex. 948. In the same figure find the projection of BC upon AC.

Ex. 949. In triangle ABC, $AC = 24$, $BC = 10$, and $\angle C = 90°$. Find the projection of AC upon AB.

[See practical applications, p. 299.]



330. NOTE. $\triangle abc$ denotes a triangle whose sides are a, b, and c. p denotes the projection of b upon c, and q the projection of a upon c. The other notations used in the following propositions are in accordance with § 244.

Ex. 950. In $\triangle abc$, if $b = 4$, $p = 2$, find $\angle A$ and h_c.

Ex. 951. In $\triangle abc$, if $b = 5$, $h_c = 4$, $c = 8$, find q.

Ex. 952. In $\triangle abc$, if $b = 10$, $h_c = 8$, and $a = 17$, find c.

Ex. 953. In $\triangle abc$, if $b = 10$, $h_c = 8$, $c = 14$, find a.

Ex. 954. In $\triangle abc$, express a in terms of b, h_c, and c.

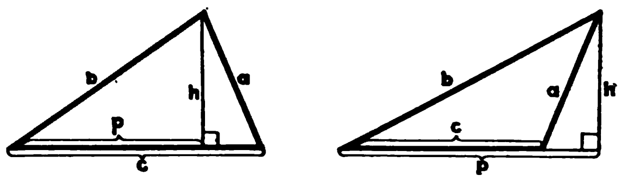
Ex. 955. In $\triangle abc$, if $a = 20$, $b = 37$, $q = 16$, find p.

Ex. 956. In $\triangle abc$, if $b = 15$, $p = 9$, and $c = 25$, find a.

Ex. 957. In $\triangle abc$, express a in terms of b, c, and p.

Proposition XXXVI. Theorem

331. *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other side upon it.*



Given in $\triangle abc$, p the projection of b upon c, and the angle opposite a an acute angle.

To prove $a^2 = b^2 + c^2 - 2\,cp.$

Proof. Denote the perpendicular upon c by h.

In the figure on the left

$$a^2 = h^2 + (c - p)^2,$$

but $h^2 = b^2 - p^2.$

[Substitute and simplify.]

In diagram II,

$$a^2 = h^2 + (p - c)^2,$$

but $h^2 = b^2 - p^2.$

[To be completed by the student.]

332. Remark. The equation $a^2 = b^2 + c^2 - 2\,cp$ contains four quantities. Therefore **any one of them may be found by algebraical methods if the other three are given.** (Similarly in the following propositions.)

Ex. 958. In $\triangle abc$, find a if

(*a*) $b = 8$, $c = 5$, $p = 4$.
(*b*) $b = 24$, $c = 9$, $p = 12$.
(*c*) $b = 5$, $c = 6$, $p = 3$.
(*d*) $b = 13$, $c = 14$, $p = 12$.
(*e*) $b = 17$, $c = 9$, $p = 15$.

Ex. 959. If two sides of a triangle equal 15 and 25 respectively, and the projection of 15 upon 25 equals 9, what is the value of the third side?

Ex. 960. In $\triangle abc$, find a if

(a) $b = 10,\ c = 16,\ \angle A = 60°$.
(b) $b = 14,\ c = 30,\ \angle A = 60°$.
(c) $b = 9,\ c = 24,\ \angle A = 60°$.
(d) $b = 48,\ c = 13,\ \angle A = 60°$.
(e) $b = 4,\ c = 3,\ \angle A = 30°$.
(f) $b = 2,\ c = 3,\ \angle A = 45°$.

Ex. 961. The sides of a triangle are 13, 14, and 15. Find the projection of 13 upon 14.

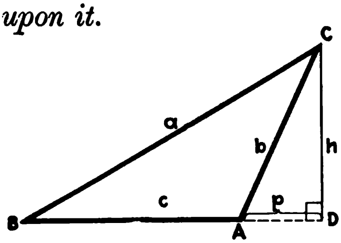
Ex. 962. The sides of a triangle are 5, 7, and 8. Find the projection of 8 upon 5.

* **Ex. 963.** The sides of a triangle are 10, 17, 21. Find the projection of 10 upon 21.

[See practical problems, pp. 298 and 299.]

PROPOSITION XXXVII. THEOREM

333. *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.*



Given in $\triangle abc$, p the projection of b upon c, and the angle opposite a obtuse.

To prove $a^2 = b^2 + c^2 + 2\,cp$.

Proof. $a^2 = h^2 + (c + p)^2$.

But $h^2 = b^2 - p^2$.

$\therefore a^2 = ?$

[To be completed by the student.]

o

Ex. 964. In $\triangle abc$, $b = 6$, $c = 10$, $p = 3$, and $\angle A$ is obtuse; find a.

Ex. 965. In $\triangle abc$, $b = 10$, $c = 9$, $p = 6$, and $\angle A$ is obtuse; find a.

Ex. 966. In $\triangle abc$, find a if

(*a*) $b = 3$, $c = 5$, $\angle A = 120°$.
(*b*) $b = 8$, $c = 7$, $\angle A = 120°$.
(*c*) $b = 16$, $c = 5$, $\angle A = 120°$.
(*d*) $b = 24$, $c = 11$, $\angle A = 120°$.

334. Remark. If we consider the projection of one side of a triangle upon another as positive when the projection lies on that line, but as negative when it lies on the prolongation, Props. XXXVII and XXXV become special cases of Prop. XXXVI, and we have always:

$$a^2 = b^2 + c^2 - 2\,cp.$$

To compute the projections of sides of a triangle whose angles are not known, always apply this equation. If the result is negative, the triangle is obtuse.

Ex. 967. In $\triangle abc$, $a = 20$, $b = 15$, and $c = 7$. Find the projection of b upon c. Is the triangle obtuse or acute?

Ex. 968. In $\triangle abc$, $a = 20$, $b = 15$, $c = 25$. Find the projection of b upon c. Is angle A obtuse or acute?

Ex. 969. The sides of a triangle are 4, 13, and 15. Find the projection of 13 upon 4.

Ex. 970. If the value of p obtained from the above formula equals zero, what does this result signify?

Ex. 971. In $\triangle abc$, $a = 15$, $b = 13$, $c = 14$. Find h_c.

Ex. 972. In $\triangle abc$, $a = 17$, $b = 10$, $c = 9$. Find h_c.

335. Cor. 1. In $\triangle abc$, if p denotes the projection of b upon c,

$$p = \frac{b^2 + c^2 - a^2}{2\,c}.$$

336. Cor. 2. If h_c denotes the altitude upon c,

$$h_c = \sqrt{b^2 - p^2} = \sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2}.$$

This expression can be simplified by algebraical operations:

$$h_c^2 = b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2 = \left(b + \frac{b^2+c^2-a^2}{2c}\right)\left(b - \frac{b^2+c^2-a^2}{2c}\right)$$

$$= \frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4c^2}$$

$$= \frac{[(b+c)^2-a^2][a^2-(b-c)^2]}{4c^2}$$

$$= \frac{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}{4c^2}.$$

Let $a+b+c=2s$, *i.e.* let s denote half the perimeter.

$$b+c-a=2(s-a).$$

$$a-b+c=2(s-b).$$

$$a+b-c=2(s-c).$$

$$h_c^2 = \frac{4s(s-a)(s-b)(s-c)}{c^2}$$

or

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

Ex. 973. In $\triangle abc$, find h_c if

(*a*) $a=10,\ b=17,\ c=21.$

(*b*) $a=20,\ b=13,\ c=21.$

(*c*) $a=20,\ b=15,\ c=25.$

(*d*) $a=37,\ b=13,\ c=40.$

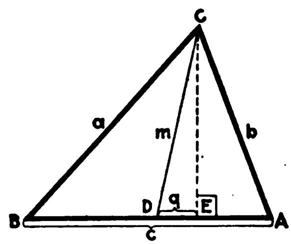
Ex. 974. The three sides of a triangle are 4, 13, and 15. Find the altitude upon 4.

Ex. 975. The three sides of a triangle are 25, 30, and 11. Find the altitude upon 11.

[See practical problems 80 to 83, p. 299.]

PROPOSITION XXXVIII. THEOREM

337. *In any triangle, the square of one side plus four times the square of the corresponding median is equal to twice the sum of the squares of the other sides.*



Given in $\triangle ABC$, m_c the median to c.

To prove $c^2 + 4\,m_c^2 = 2\,a^2 + 2\,b^2$.

Proof. Draw $CE \perp AB$, and suppose E to fall between A and D. Let $DE = q$.

$$a^2 = \left(\frac{c}{2}\right)^2 + m^2 + 2\left(\frac{c}{2}\right)q. \qquad (333)$$

$$b^2 = \left(\frac{c}{2}\right)^2 + m^2 - 2\left(\frac{c}{2}\right)q. \qquad (331)$$

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\,m^2, \qquad \text{(Ax. 2.)}$$

or

$$2\,a^2 + 2\,b^2 = c^2 + 4\,m^2. \qquad \text{(Ax. 7.)}$$

Q. E. D.

Ex. 976. The sides of a triangle are 7, 8, and 9 respectively. Find the length of the median to 8.

Ex. 977. The sides of a triangle are 7, 4, and 9 respectively. Find the length of the median to 9.

Ex 978. The sides of a triangle are 10, 5, and 9 respectively, Find the length of the median to 9.

Ex. 979. The sides of a triangle are 22, 20, and 18 respectively. Find the length of the median to 18.

Ex. 980. In $\triangle abc$, if m_c denotes the median drawn to c, prove that

$$m_c = \frac{1}{2}\sqrt{2\,a^2 + 2\,b^2 - c^2}.$$

Ex. 981. In $\triangle abc$, $a = 8$, $b = 11$, and $m_c = 8\frac{1}{2}$. Find c.

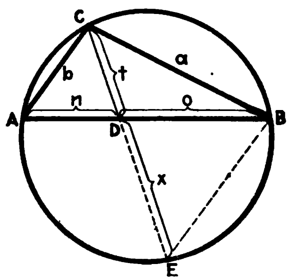
Ex. 982. In $\triangle abc$, $a = 28$, $c = 32$, $m_c = 38$. Find b.

Ex. 983. Prove proposition XXXVIII if $a = b$.

Ex. 984. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

Proposition XXXIX. Theorem

338. *In any triangle, the product of two sides is equal to the square of the bisector of the included angle plus the product of the segments of the third side.*



Given $\triangle abc$, the angle = bisector t dividing c into the segments n and o.

To prove $ab = t^2 + on.$

Proof. Circumscribe a circle about $\triangle abc$.

Produce the angle = bisector CD to meet the circumference in E. Draw EB, and let $DE = x$.

$\angle ACD = \angle ECB.$ (Hyp.)

$\angle A = \angle E.$ (233)

$\triangle ACD \sim \triangle EBC.$ (305)

$\therefore b : t + x = t : a,$

or $ab = t(t + x).$

$ab = t^2 + tx.$

But $tx = on.$ (320)

$\therefore ab = t^2 + on.$ Q. E. D.

Ex. 985. The sides of a triangle are 18, 9, and 21 respectively. Find the length of the bisector corresponding with 21.

HINT. Find n and o by means of (301).

Ex. 986. The sides of a triangle are 21, 14, and 25 respectively. Find the length of the bisector corresponding with 25.

Ex. 987. The sides of a triangle are 22, 11, and 21 respectively. Find the length of the bisector corresponding with 21.

Ex. 988. The sides of a triangle are 6, 3, and 7 respectively. Find the length of the bisector corresponding with 7.

Ex. 989. In $\triangle\, abc$, if t denotes the angle bisector of $\angle\, C$, prove that

$$t^2 = ab - \frac{abc^2}{(a+b)^2}.$$

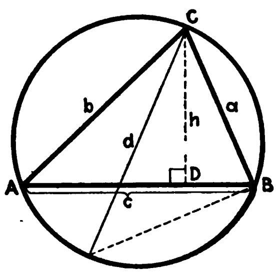
HINT. $$n = \frac{bc}{a+b},\ o = \frac{ac}{a+b}. \qquad (301)$$

* **Ex. 990.** Using the notation and the method of (336), reduce the result of the preceding exercise to the following form

$$t = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

PROPOSITION XL. THEOREM

339. *In any triangle the product of two sides is equal to the altitude upon the third side, multiplied by the diameter of the circumscribed circle.*



Given d, a diameter of the circle circumscribed about $\triangle\, abc$, and h, the altitude upon c.

To prove $ab = hd.$

[The proof is left to the student.]

340. Cor. The diameter of the circumscribed circle of any triangle is equal to the product of two sides divided by the altitude upon the third side.

$$\left(d = \frac{ab}{h} \text{ or } d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}\right).$$

Ex. 991. In $\triangle abc$, $a = 6$, $b = 10$, and $h_c = 4$; find the diameter of the circumscribed circle.

Ex. 992. In $\triangle abc$, $a = 10$, $b = 15$, $h_c = 6$. Find the radius of the circumscribed circle.

Ex. 993. $\triangle ABC$ is inscribed in a circle of radius $= 5$ inches. Find the altitude to BC if $AB = 4$, and $AC = 5$.

Ex. 994. Find the diameter of the circle circumscribed about $\triangle abc$, if

(*a*) $a = 17$, $b = 8$, $c = 15$.
(*b*) $a = 10$, $b = 17$, $c = 21$.

Ex. 995. In $\triangle abc$, $a = 20$, $b = 15$, and the projection of b upon c equals 9. Find the radius of the circumscribed circle.

Ex. 996. In $\triangle abc$, $a = 9$ and $b = 12$. Find c if the diameter of the circumscribed circle equals 15.

[See practical applications, p. 300.]

PROBLEMS OF COMPUTATION

Ex. 997. The arms of a right triangle are 8 and 15 respectively. Compute the hypotenuse and the altitude upon the hypotenuse.

Ex. 998. In $\triangle abc$, $a = 9$, $b = 15$, and $c = 17$. Is the triangle obtuse, right, or acute?

Ex. 999. The arms of a right triangle are $m^2 - n^2$ and $2mn$ respectively. Find the hypotenuse.

Ex. 1000. A chord 14 in. long is 12 in. distant from the center. Find the radius of the circle.

Ex. 1001. A chord 24 in. long is 5 in. distant from the center. Find the distance of the center from a chord 10 in. long.

Ex. 1002. In $\triangle abc$, $a = 3$, $b = 8$, and the angle opposite to c equals $60°$. Find c.

Ex. 1003. A straight line $AB = 4$, is divided externally in the ratio $5:4$. Find the segments.

Ex. 1004. The shadow of a church steeple upon level ground is 60 ft., while a pole 10 ft. high casts a shadow 8 ft. long. How high is the steeple?

Ex. 1005. Find the product of the segments of a chord drawn through a point 8 in. from the center of a circle whose radius is 10 in. What is the length of the shortest chord that can be drawn through that point?

Ex. 1006. In $\triangle abc$, $b = 13$, $c = 35$, and the angle opposite to a equals 120°. Find c.

Ex. 1007. In $\triangle abc$, $a = 10$, $b = 17$, $c = 21$. Find the altitude upon 10.

Ex. 1008. The line of centers of two circles is equal to 10. Find the length of the common chord if the radii are 8 and 6 respectively.

Ex. 1009. The base of an isosceles triangle is 48 in. Find the altitude if each arm equals 50 in.

Ex. 1010. Two sides and a diagonal of a parallelogram are 7, 9, and 8 respectively; find the length of the other diagonal.

Ex. 1011. The diagonal of a square is 20 in. Find the side.

Ex. 1012. The sides of a rectangle are 16 and 30 respectively. Find the diagonal.

Ex. 1013. The diameter AB of a circle is produced to C, and from C a tangent is drawn to the circle. Find the length of the tangent if $AB = 30$ and $BC = 2$.

Ex. 1014. In $\triangle abc$, $a = 16$, $b = 18$, and $c = 22$. Find the median to b.

Ex. 1015. The base of an isosceles triangle is 4, and the arm 7. Find the median to one of the arms.

Ex. 1016. A ladder 17 ft. long reaches a window 15 ft. high. How far is the lower end of the ladder from the house?

Ex. 1017. In $\triangle abc$, $a = 18$, $b = 23$, and $c = 9$. Find the bisector of the angle opposite b.

Ex. 1018. In $\triangle abc$, $c = 10$, $\angle A = 30°$, and $\angle C = 90°$. Find a and b.

Ex. 1019. The diagonals of a parallelogram are 30 and 26 in. Find the altitude if the base equals 14 in.

Ex. 1020. In $\triangle abc$, $a = 17$, $b = 10$, $c = 21$. Find the radius of the circumscribed circle.

Ex. 1021. The base of an isosceles triangle is b, and each arm a. Find the altitude.

Ex. 1022. The non-parallel sides AB and CD of a trapezoid are produced till they meet in E. Find AE and BE if $AB = 7$ and the bases are 5 and 3 respectively.

Ex. 1023. The altitude of a trapezoid is h, the bases a and b respectively. Find the altitudes of the two triangles formed by producing the non-parallel sides until they meet.

Ex. 1024. From a point 24 ft. above sea level the visible horizon has a radius of 6 miles. Find the diameter of the earth.

Ex. 1025. Find the length of the common internal tangent of two circles if the line of centers is 17, and the radii are 5 and 3 respectively.

Ex. 1026. A chord 30 in. long subtends an angle of 120°. Find the distance of the chord from the center of the circle.

* **Ex. 1027.** In triangle ABC, $AB = BC = 25$, $AC = 30$, and on AB is laid off $AD = 8$. Find the length of CD.

* **Ex. 1028.** The three sides of a triangle are 14, 16, and 6. Find the angle opposite 14.

* **Ex. 1029.** In a quadrilateral $ABCD$, $AB = 10$, $BC = 17$, $CD = 13$, $DA = 20$, and $AC = 21$. Find the diagonal BD.

PROBLEMS OF CONSTRUCTION

Ex. 1030. Divide any side of a triangle into two parts proportional to the other two sides.

To construct a triangle, having given : (230)

Ex. 1031. a, b, and $b : c = 4 : 5$.

Ex. 1032. a, $b + c$, and $b : c = 3 : 4$.

Ex. 1033. From a given rectangle to cut off a similar rectangle by a line parallel to one of its sides.

Ex. 1034. In a given circle, to inscribe a triangle similar to a given triangle.

Ex. 1035. About a given circle, to circumscribe a triangle, similar to a given triangle.

Ex. 1036. Construct a triangle similar to a given triangle and having a given altitude.

Ex. 1037. To inscribe a square in a given triangle.

Ex. 1038. Assuming an arbitrary unit construct a line equal to (a) $\sqrt{2}$, (b) $\sqrt{3}$, (c) $1 + \sqrt{5}$.

Ex. 1039. Construct a line that shall be to a given line as $1 : \sqrt{2}$.

Ex. 1040. Construct a line that shall be to a given line as $\sqrt{3} : 1$.

* **Ex. 1041.** Construct a triangle similar to a given triangle and having a given median.

Ex. 1042. In a given line AB, to find a point C such that $AC : BC = 1 : \sqrt{2}$.

Ex. 1043. To construct a parallelogram similar to a given parallelogram and having a given diagonal.

Ex. 1044. To construct a triangle similar to a given triangle and having a given perimeter.

Ex. 1045. If a and b are two given lines, construct a line equal to $\frac{a^2}{b}$.

Ex. 1046. From a point without a circle, to draw a secant whose external segment is equal to one half the secant.

Ex. 1047. To construct a circle, touching a given circle in a given point, and touching a given line. HINT. Draw a tangent to the circle.

Ex. 1048. To construct a circle, touching two parallel lines and passing through a given point.

Ex. 1049. In a given circle, to inscribe a rectangle, having given the ratio of two sides.

* **Ex.** 1050. To divide a trapezoid into two similar trapezoids by a line parallel to the base.

Ex. 1051. In the prolongation of the side AB of the triangle ABC to find a point X such that $AX \times BX = \overline{CX}^2$. (322.)

Ex. 1052. Through a given point P, to draw a line such that its distances from two other given points, R and S, shall have a given ratio.

* **Ex.** 1053. Through a given point within a circle, to draw a chord so that its segments have a given ratio.

* **Ex.** 1054. Through a point of intersection of two circles, to draw a line forming equal chords.

* **Ex.** 1055. To construct two lines, having given their mean proportional and their difference.

THEOREMS

Ex. 1056. If a chord is bisected by another, either segment of the first is a mean proportional between the segments of the other.

Ex. 1057. If in the triangle ABC the altitudes BD and AE meet in F, and $AB = BC$, then

$$BC : AF = BD : CD.$$

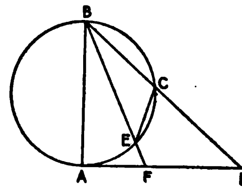
Ex. 1058. Two triangles are similar if an angle of the one is equal to an angle of the other, and the altitudes corresponding with the other angles are proportional.

Ex. 1059. If between two parallel tangents, a third tangent is drawn, the radius is the mean proportional between the segments of the third tangent.

Ex. 1060. If two circles are tangent externally, and through the point of contact a secant is drawn, the chords formed are proportional to the radii.

Ex. 1061. If C is the mid-point of the arc AB, and a chord CD meets the chord AB in E, then

$$CE : CA = CA : CD.$$



Ex. 1062. In the annexed diagram, if AB is a diameter, AD a tangent, and FB and DB are secants, prove that

$$BE \times BF = BC \times BD.$$

Ex. 1063. Prove that in any right triangle the sum of the hypotenuse and the diameter of the inscribed circle is equal to the sum of the legs of the triangle.

Ex. 1064. If two circles intersect, their common chord produced bisects the common tangents.

Ex. 1065. If an isosceles triangle is inscribed in a circle, the tangents drawn at the vertices form another isosceles triangle.

Ex. 1066. The tangents drawn at the vertices of an inscribed rectangle inclose a rhombus.

Ex. 1067. Two parallelograms are similar when they have an angle of the one equal to an angle of the other, and the including sides proportional.

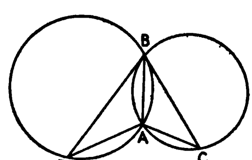
Ex. 1068. Two rectangles are similar if two adjacent sides are proportional.

Ex. 1069. Three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians.

Ex. 1070. If in rectangle $ABCD$ a perpendicular is drawn from D upon AC, the prolongation of which intersects AB in E, then

$$AE : AD = AD : CD.$$

Ex. 1071. Two circles are tangent externally, and through the point of contact two straight lines are drawn terminating in the circumferences; prove that the corresponding segments of the lines are proportional.



Ex. 1072. If two circles intersect in A and B and the two chords BD and BC are respectively tangents to the two circles, AB is the mean proportional between AD and AC.

Ex. 1073. If two circles are tangent internally, chords of the greater circle drawn from the point of contact are divided proportionally.

* **Ex. 1074.** If in a triangle the squares of two unequal sides have the same ratio as their projections upon the third side, the triangle is a right triangle.

* **Ex. 1075.** If from a point O, OA, OB, OC, and OD are drawn so that the angle AOB is equal to the angle BOC, and the angle BOD equal to a right angle, any line intersecting OA, OB, OC, and OD is divided harmonically. (301, 302.)

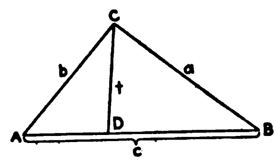
* **Ex. 1076.** The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the mid-points of the diagonals. (337.)

* **Ex. 1077.** If from a point within the triangle ABC the perpendiculars OX, OY, and OZ be drawn upon AB, BC, and CA respectively,

$$\overline{AX}^2 + \overline{BY}^2 + \overline{CZ}^2 = \overline{BX}^2 + \overline{YC}^2 + \overline{ZA}^2.$$

* **Ex. 1078.** State and prove the converse of the preceding exercise.

* **Ex. 1079.** If two circles are tangent externally, either common external tangent is a mean proportional between the diameters.



* **Ex. 1080.** If in $\triangle ABC$ point D is taken in AB such that $AD : DB = 1 : 2$, and $CD = t$, prove that

$$9\,t^2 + 2\,c^2 = 6\,b^2 + 3\,a^2.$$

BOOK IV

AREAS OF POLYGONS

341. DEF. The **unit of surface** is a square whose side is the unit of length, etc.

Thus, a square 1 in. long and 1 in. wide is a square inch. Or a square 1 yd. long and 1 yd. wide is a square yard.

342. The **area of a surface** is the number of units of surface it contains.

Thus, if the floor of a room is 25 ft. long and 15 ft. wide, it contains 15×25 or 375 sq. ft. Hence the area of the floor is 375 sq. ft.

343. Two figures are **equivalent** or **equal** if their areas are equal.

Thus, if the area of $\triangle ABC = 25$ sq. ft., and the area of $\square MNOP = 25$ sq. ft., then $\triangle ABC$ is *equivalent to* $\square MNOP$, or in symbols:

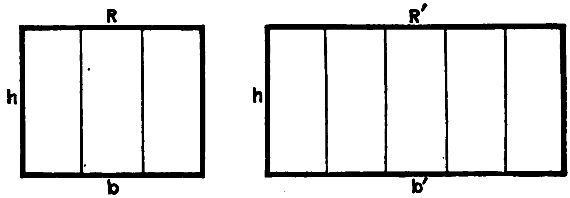
$$\triangle ABC = \square MNOP.$$

NOTE. The symbol $=$ refers to the size of figures, while the symbol $\sim$ relates to their shape. The symbol of congruence ($\cong$) relates to both size and shape. The symbol $\sim$ has no meaning for figures that cannot differ in shape, *e.g.* for straight lines. Thence the symbol of congruence ($\cong$) when applied to straight lines has the same significance as the symbol of equality ($=$).

If the symbol of equality ($=$) refers to areas, it may read either "is equivalent to" or "equals." Since many authors, however, designate congruent figures as *equal figures*, confusion may be avoided by giving preference to the term equivalent. The use of a particular symbol for equivalence ($\Leftrightarrow$) cannot be recommended.

PROPOSITION I. THEOREM

344. *Rectangles having equal altitudes are to each other as their bases.*



Given R and R' the areas of two rectangles whose common altitude equals h, and whose bases are respectively b and b'.

To prove $\frac{R}{R'}=\frac{b}{b'}.$

Proof. CASE I. $\frac{b}{b'}$ is a rational number. Let $\frac{b}{b'}=\frac{m}{n}$; *i.e.* if b' is divided into n equal parts, and one of these parts is laid off on b, b contains m such parts.

If perpendiculars are erected at the points of division, R is divided into n rectangles and R' is divided into m rectangles.

These rectangles are all equal. (150)

$$\therefore \frac{R}{R'}=\frac{m}{n}.$$

$$\therefore \frac{R}{R'}=\frac{b}{b'}. \qquad \text{(Ax. 1.)}$$

CASE II. $\frac{b}{b'}$ is an irrational number. Since any approximate value of $\frac{b}{b'}$ is a rational number, it must equal the corresponding approximate value of $\frac{R}{R'}$. (Case I.) Hence all corresponding approximate values of $\frac{b}{b'}$ and $\frac{R}{R'}$ are equal.

$$\therefore \frac{R}{R'}=\frac{b}{b'}. \qquad (223)$$

345. Cor. Rectangles having equal bases are to each other as their altitudes.

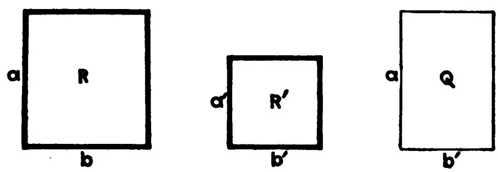
Ex. 1081. To divide a given rectangle into three equivalent parts.

Ex. 1082. From a given rectangle, to cut off another rectangle whose area is $\frac{3}{5}$ of the given one.

Ex. 1083. To construct a rectangle, which is to a given one as $m : n$, when m and n are two given lines.

Proposition II. Theorem

346. *The areas of two rectangles are to each other as the products of their bases and altitudes.*



Given rectangles R and R' having the bases b and b' and the altitudes a and a' respectively.

To prove $$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

Proof. Construct the rectangle Q, having the same base as R', and the same altitude as R.

$$\frac{R}{Q} = \frac{b}{b'}. \qquad (344)$$

$$\frac{Q}{R'} = \frac{a}{a'}. \qquad (345)$$

Multiplying member by member,

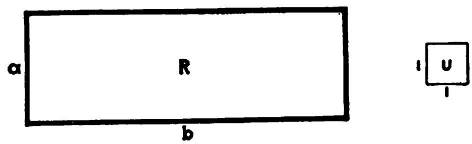
$$\frac{R}{R'} = \frac{a \times b}{a' \times b'}. \qquad \text{Q. E. D.}$$

Ex. 1084. Find the ratio of a rectangle 4 by 5 ft., and a square having a side of 10 ft.

Ex. 1085. The diagonal of a rectangle is 26 in. long, and one of its sides 24 in. The diagonal of another rectangle is 25 in. long, and one of its sides 20 in. Find the ratio of the areas of the two rectangles.

PROPOSITION III. THEOREM

347. *The area of a rectangle is equal to the product of its base and altitude.*



Given R a rectangle with base b and altitude a.

To prove $R = a \times b$.

Proof. Let U be the unit of surface.

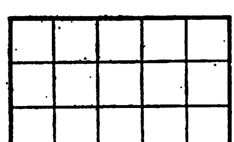
Then $\frac{R}{U} = \frac{a \times b}{1 \times 1}$. (346)

But $\frac{R}{U}$ is the area of R. (342)

$\therefore R = a \times b$. Q. E. D.

348. COR. The area of a square is equal to the square of its side.

349. REMARK. If the base and altitude of a rectangle are expressed by integral numbers, Prop. III may be proved by dividing the figure into squares.

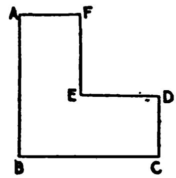


Thus, if the base contains 5 and the altitude 3 linear units, the figure can be divided into fifteen squares, each being the unit of surface.

Ex. 1086. A rectangular field is 24 yd. long and 15 yd. wide. Find the area.

Ex. 1087. The area of a rectangle is 360 sq. in., and its base is 4 yd. Find the altitude.

Ex. 1088. A rectangle has an area of 125 sq. in. and is five times as long as wide. Find the dimensions of the rectangle.



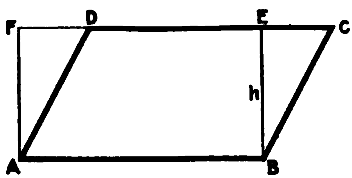
Ex. 1089. In the annexed diagram, all angles (A, B, C, etc.) are rt. angles. Find the area of the figure if

(*a*) $AB = 9$, $BC = 8$, $DE = 4$, $EF = 3$.

(*b*) $AB = BC = a$, $FE = ED = b$.

Proposition IV. Theorem

350. *The area of a parallelogram is equal to the product of its base and its altitude.*



Given parallelogram $ABCD$ having the base $AB = b$, and the altitude $BE = h$.

To prove $ABCD = b \times h$.

Proof. Draw $AF \perp AB$, meeting CD produced in F.

Then $ABEF$ is a rectangle, having its base b and its altitude h.

$ABCF = ABCF$, (Iden.)

But $\triangle AFD = \triangle BEC$,* (hy. arm = hy. arm.)

Subtracting member from member,

$ABCD = ABEF$.

But $ABEF = b \times h$. (347)

$\therefore ABCD = b \times h$. Q. E. D.

* Obviously figures that are congruent must also be equivalent.

P

351. Cor. 1. Parallelograms having equal bases and equal altitudes are equivalent.

352. Cor. 2. Any two parallelograms are to each other as the products of their bases and altitudes.

353. Cor. 3. Parallelograms having equal bases are to each other as their altitudes.

354. Cor. 4. Parallelograms having equal altitudes are to each other as their bases.

Ex. 1090. Find the area of a parallelogram whose base is 15 in. and whose altitude is 2 ft.

Ex. 1091. Two sides of a parallelogram are 15 and 20 respectively, and include an angle of 30°. Find the area.

Ex. 1092. The sides of a parallelogram are 5 and 8 respectively, and the projection of 5 upon 8 is three. Find the area.

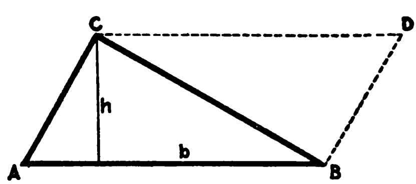
Ex. 1093. To divide a parallelogram into three equivalent parts.

Ex. 1094. The sides of a parallelogram are 13 and 14, and one diagonal equals 15. Find the area.

Ex. 1095. The sides of a parallelogram are 4 in. and 5 in. and the angle included by them equal 45°. Find the area.

Proposition V. Theorem

355. *The area of a triangle is equal to one half the product of its base and altitude.*



Given $\triangle ABC$, having base b and altitude h.

To prove $\triangle ABC = \frac{1}{2} b \times h$.

Proof. Construct $\square$ $ABDC$.

The diagonal of a parallelogram divides it into two equal triangles.

$$\therefore \triangle ABC = \tfrac{1}{2} \square ABDC. \quad (142)$$

But $$\square ABDC = b \times h. \quad (350)$$

$$\therefore \triangle ABC = \tfrac{1}{2} b \times h. \quad \text{(Sub.)}$$

Q. E. D.

356. Cor. 1. Triangles having equal bases and equal altitudes are equivalent.

357. Cor. 2. Any two triangles are to each other as the products of their bases and altitudes.

358. Cor. 3. Triangles having equal bases are to each other as their altitudes.

359. Cor. 4. Triangles having equal altitudes are to each other as their bases.

360. Cor. 5. If two triangles have a common base and their vertices lie in a line parallel to the base, the triangles are equivalent.

361. Cor. 6. If A denotes the area, and $2s$ the perimeter of $\triangle abc$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad (336)$$

362. Def. To **transform a figure** means to construct another figure equivalent to the given figure.

Ex. 1096. What is the locus of the vertices of all the equivalent triangles constructed on the same base?

Ex. 1097. To transform a given triangle into a right triangle.

Ex. 1098. To transform $\triangle ABC$ into an isosceles triangle, having its base equal to AB.

Ex. 1099. To transform $\triangle ABC$ into an isosceles triangle, having an arm equal to AB.

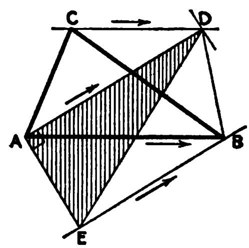
Ex. 1100. To transform a given triangle into another one, having given one side.

Ex. 1101. To transform a given triangle into a right triangle, having the hypotenuse equal to one side of the given triangle.

Ex. 1102. To transform a given triangle ABC into another one, having its base equal to AB and its vertical angle equal to a given angle. (255)

363. Method XX. **The fundamental method for transforming triangles is based upon Cor. 5 of Prop. V.**

Since by this transformation one side (the base) remains unaltered, it may be applied twice in succession, provided that the first transformation makes one side equal to a given line, and that in the second transformation this side is made the base of the triangle.



Thus, to transform a triangle ABC into a right triangle having one arm equal to a given line m, first transform $\triangle ABC$ into $\triangle ABD$, having $AD = m$. Then consider AD as base, and transform $\triangle ABD$ into rt. $\triangle AED$, which is the required one.

Note. In the problems of transformation which follow, the given figure is drawn in heavy lines, and the resulting figure is shaded.

Ex. 1103. To transform $\triangle ABC$ into another triangle, having two of its sides respectively equal to two given lines m and n.

Ex. 1104. To transform $\triangle ABC$ into another triangle, having a side equal to a given line m, and one adjacent angle equal to a given angle A.

Ex. 1105. To transform $\triangle ABC$ into a right triangle, having the hypotenuse equal to a given line m.

Ex. 1106. To transform $\triangle ABC$ into an isosceles triangle, having the base equal to a given line m.

Ex. 1107. To transform $\triangle ABC$ into an isosceles triangle, having an arm equal to a given line m.

Ex. 1108. To transform $\triangle ABC$ into a triangle having one side equal to a given line m, and the opposite angle equal to a given angle Q.

Ex. 1109. To transform a parallelogram into a rectangle.

Ex. 1110. To transform a parallelogram into a rhombus.

Ex. 1111. To transform a parallelogram into another one having a given side.

Ex. 1112. To transform a parallelogram into another one containing a given angle.

Ex. 1113. To transform a parallelogram into another parallelogram having two of its sides respectively equal to two given lines.

Ex. 1114. To transform a parallelogram into another parallelogram having one side equal to a given line and one angle equal to a given angle.

Ex. 1115. To transform a parallelogram into a rectangle, having one side equal to a given line.

Ex. 1116. To divide a triangle ABC into three equivalent parts by two lines passing through A.

Ex. 1117. To divide a given parallelogram into two equivalent parts by a line parallel to the bases.

Ex. 1118. To divide a given parallelogram into two equivalent parts by a line perpendicular to the bases.

Ex. 1119. To divide a given parallelogram into two equivalent parts by a line parallel to a given line.

Ex. 1120. Transform quadrilateral $ABCD$ into quadrilateral $ABCE$ so that the vertices A, B, and C remain unaltered, and

(*a*) CE equals a given line.
(*b*) $\angle BCE$ equals a given angle.
(*c*) BE equals a given line.
(*d*) $CE = EA$.
(*e*) $\angle BCE = 180°$.

Ex. 1121. To transform a given quadrilateral into a triangle.

Ex. 1122. To transform $\triangle ABC$ into a triangle containing the angle A, and having one vertex in D, if D is a given point in AC.

HINT. Consider $ABCD$ a quadrilateral, and apply Ex. 1121.

Ex. 1123. The diagonals divide a parallelogram into four equivalent triangles.

Ex. 1124. Two triangles are equivalent if two sides of the one are respectively equal to two sides of the other, and the included angles are supplementary.

Ex. 1125. To construct a triangle three times as large as a given triangle.

Ex. 1126. The lines joining the mid-point of a diagonal of a quadrilateral with the opposite vertices divide the figure into two equivalent parts.

Ex. 1127. To divide a quadrilateral into three equivalent parts.

Ex. 1128. The area of a triangle is 600 sq. in., and the altitude is 20 in. Find the base.

Ex. 1129. Two sides of a triangle are 5 and 8 respectively, and include an angle of 30°. Find the area.

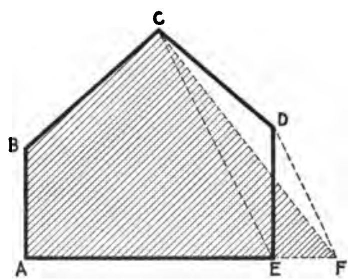
Ex. 1130. Find the area of a triangle whose sides are respectively

(*a*) 13, 14, 15.
(*b*) 9, 10, 17.
(*c*) 11, 25, 30.
(*d*) 4, 13, 15.

Ex. 1131. Find the area of $\triangle\, abc$, if $a = 10$, $b = 24$, $m_c = 13$.

Proposition VI. Problem

364. *To transform a polygon into a triangle.*



Given polygon *ABCDE*.

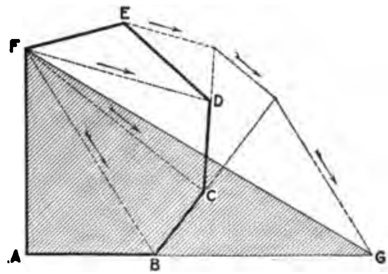
Required to construct a triangle equivalent to *ABCDE*.

Construction. Draw *CE*.

Through *D*, draw $DF \parallel CE$, meeting *AE* produced in *F*.
Then polygon $ABCF = ABCDE$, and has one less side.
Continue the process until a triangle is obtained.

[The proof is left to the student.]

365. Remark. In a complete representation of this construction the student is advised to draw the diagonals (FD, FC, FB) from the *same* vertex (F) and to omit all lines that are needed for the proof only. The annexed figure represents the construction, thus obtained, for a hexagon.

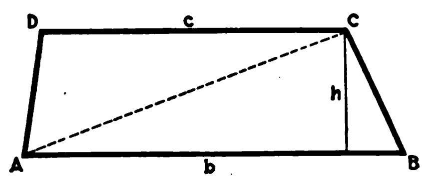


Ex. 1132. To transform a parallelogram into a triangle.

Ex. 1133. To transform a pentagon into a right triangle.

Proposition VII. Theorem

366. *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*



Given. Trapezoid $ABCD$ has the bases b and c respectively, and the altitude h.

To prove $ABCD = \frac{1}{2} h (b + c)$.

Proof. Draw AC.

$\triangle ABC = \frac{1}{2} h \times b$. (355)

$\triangle DCA = \frac{1}{2} h \times c$. (355)

$\therefore ABCD = \frac{1}{2} h (b + c)$. Q. E. D.

Ex. 1134. The area of a trapezoid equals the product of its altitude and its median.

Ex. 1135. A line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equivalent parts.

Ex. 1136. The bases of a trapezoid are 12 and 8 respectively, and the altitude is 5. Find the area.

Ex. 1137. In the diagram for Prop. VII, if $b = 6$, $c = 4$, $BC = 10$, and $\angle B = 30°$. Find the area.

Ex. 1138. The area of a trapezoid is 200, the bases are 15 and 25 respectively. Find the altitude.

Ex. 1139. The area of a trapezoid is 30, the altitude is 5, and one base is 8. Find the other base.

Ex. 1140. In the diagram for Prop. VII, $b = 14$, $c = 10$, $AC = 15$, and $BC = 13$. Find the area.

Ex. 1141. In the same diagram $b = 8$, $c = 6$, $BC = 13$, and the projection of BC upon AB equals 5. Find the area.

Ex. 1142. In the diagram for Prop. VII if E, the mid-point of DA, be joined to C and B, $\triangle CBE$ is one half of $ABCD$.

Ex. 1143. If the three altitudes of a triangle are equal, the triangle is equilateral.

367. The product of (the numerical measures of) two lines can be represented geometrically by the area of the rectangle formed by these lines. Hence, if the letters a, b, c, etc., represent a number of lines, then any homogeneous expression of the second degree that involves these letters can be represented by the sum (or difference) of rectangles.

NOTE. All diagrams of the following exercises may be constructed by drawing first two lines that are perpendicular to each other, and then constructing parallels to these according to the conditions of the theorem.

We may therefore assume in each exercise the following fact, which can easily be proved:

All quadrilaterals in the diagram are rectangles, and their opposite sides are equal.

Ex. 1144. If a, b, c, and d are four lines, $a > b$, and $b > c$ represent geometrically

(a) ab. (c) $(a - c)d$. (e) $(a - b)(c + d)$.

(b) $(a + b)b$. (d) $(a + b)(c + d)$. (f) $ab + ac$.

(*g*) $a^2 + ab + b^2$. (*k*) $a^2 - b^2$.
(*h*) $a(a + b) + b(a + b)$. (*l*) $(a + b)^2$.
(*i*) $(a + b)(a - b)$. (*m*) $(a - b)^2$.

368. Any homogeneous identity of the second degree can be demonstrated geometrically by showing that the areas which represent the two members of the identity are equal.

Thus to illustrate that

$(a + b)c = ac + bc$ we have,

$(a + b)c =$ area of AB.

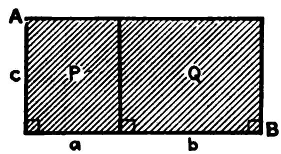
$ac =$ area of P.

$bc =$ area of Q.

But obviously

$$AB = P + Q.$$

or $(a + b)c = ac + bc$. Q. E. D.



Similarly to show that

$(a - c)(b + d) = ab + ad - bc - bd$.

$AH = a - c$, $AC = b + d$.

$\therefore$ the left member of one identity, or

$(a - c)(b + d) =$ area of AD.*

But $a \cdot b = GB$, $a \cdot d = FC$.

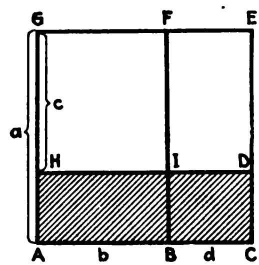
$\therefore ab + ad = GC$,

and $bc = HF$, $dc = FD$.

$\therefore ab + ad - bc - dc = AD$,

or $(a - c)(b + d) = ab + ad - bc - dc$.

Q. E. D.



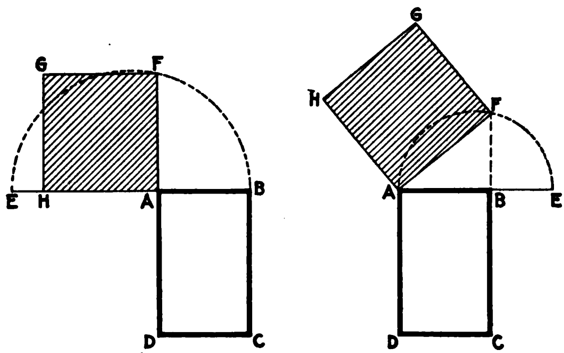
Ex. 1145. Prove geometrically the following algebraic formulæ:

(*a*) $a(b + c + d) = ab + ac + bd$.
(*b*) $(a + b)(c + d) = ac + bc + ad + bd$.
(*c*) $a(b - c) = ab - ac$.
(*d*) $a(b + c - d) = ab + ac - ad$.
(*e*) $(a + b)^2 = a^2 + 2\,ab + b^2$.
(*f*) $a^2 - b^2 = (a + b)(a - b)$.
(*g*) $(a - b)^2 = a^2 - 2\,ab + b^2$. (Hint. $a^2 - ab + b^2 - ab$.)
(*h*) $(a + b)^2 - (a - b)^2 = 4\,ab$.

* The student is advised in the following exercises to shade the area which represents the left member.

PROPOSITION VIII. PROBLEM

369. *To transform a rectangle into a square.*



Given rectangle $ABCD$.

Required a square $= ABCD$.

Construction. Construct a mean proportional AF between AB and AD. The square on AF is the required square.

Proof. $AB : AF = AF : AD$. (Con.)

$\therefore AB \times AD = \overline{AF}^2$, (276)

or area $ABCD =$ area AG. Q.E.F.

370. REMARK. The mean proportional may be constructed by any of the methods of Book III. The two indicated in the diagram, however, are the most useful ones.

Ex. 1146. To transform a parallelogram into a square.

Ex. 1147. To transform a triangle into a parallelogram.

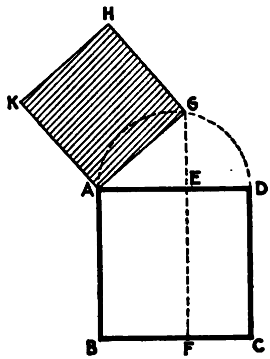
Ex. 1148. To transform a triangle into a square.

Ex. 1149. To transform a pentagon into a square.

Ex. 1150. To construct a square equivalent to one third of a given triangle.

Proposition IX. Problem

371. *To construct a square that shall be any given part of a given square.*



Given square AC.

Required a square equivalent to $\frac{4}{7}$ of AC.

Construction. On AD, lay off $AE = \frac{4}{7}\,AD$.

Draw $EF \perp AD$, and transform rectangle $AEFB$ into a square, AH. (369)

AH is the required square.

Proof. $AH = AF$ (Con.)

$AF = \frac{4}{7}\,AC.$ (344)

$\therefore\ AH = \frac{4}{7}\,AC.$ Q. E. F.

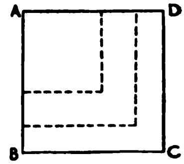
372. Remark. If, instead of $\frac{4}{7}$, the ratio $\frac{m}{n}$ is given, where m and n are two given lines, make AE the fourth proportional to n, m, and AD.

Ex. 1151. To construct a square three times as large as a given square.

Ex. 1152. Given a line AB. To construct another line whose square equals $\frac{3}{5}$ of the square of AB.

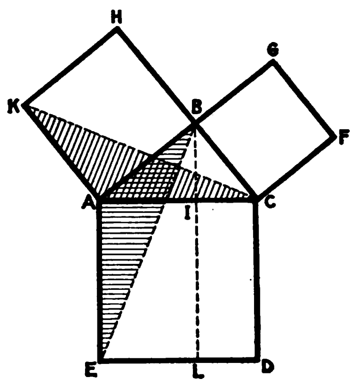
Ex. 1153. Given a line AB. To construct another line whose square equals twice the square of AB.

Ex. 1154. To divide a given square $ABCD$ into three equivalent parts by constructing two other squares which contain angle A.



PROPOSITION X. THEOREM

373. *In a right triangle the sum of the squares on the arms is equivalent to the square on the hypotenuse.**



Given AH and BF squares on the arms, AD the square on the hypotenuse, of the right triangle ABC.

To prove $AD = AH + BF$.

Proof. From B draw $BL \parallel AE$, intersecting AC and ED in I and L respectively.

Draw BE and KC.

$$\angle HBC = \text{st. } \angle,$$

or HBC is a straight line.

* The shading in this figure is employed merely to point out the two congruent triangles.

In $\triangle$ AKC and BAE,

$KA = AB$ and $AC = AE$.

$\angle BAC = \angle BAC$. (Why?)

$\angle KAB = \angle CAE$. (Why?)

$\therefore \angle KAC = \angle BAE$. (Why?)

$\therefore \triangle KAC \cong \triangle BAE$. (Why?)

But square $AH = 2 \triangle KAC$, (Why?)

and rectangle $EI = 2 \triangle BAE$.

$\therefore$ square $AH =$ rectangle EI.

In like manner,

square $BF =$ rectangle CL.

$\therefore$ square $AH +$ square $BF =$ square AD, Q. E. D.

374. Cor. The square on an arm of a right triangle is equivalent to the difference between the squares on the hypotenuse and the other arm.

375. Note. The above theorem was first demonstrated by Pythagoras about 550 B.C., and is called after him the **Pythagorean Theorem**.

The proof given here was given by Euclid (about 300 B.C.).

Since a square on a line is measured by the square of the line, this theorem may also be demonstrated by the proof in 326, Book III.

Proposition XI. Problem

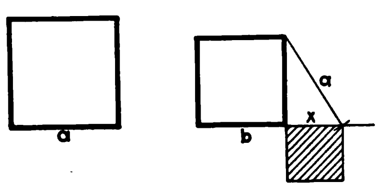
376. *To construct a square equivalent to the sum of two given squares.*

[The solution is left to the student.]

Proposition XII. Problem

377. *To construct a square equivalent to the difference of two given squares.*

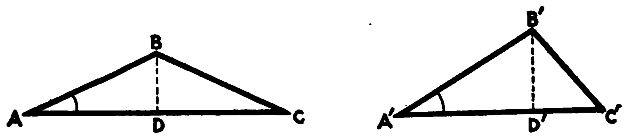
[The solution is left to the student.]



Ex. 1155. To construct a square equivalent to the sum of three given squares.

Proposition XIII. Theorem

378. *The areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the product of the sides including the equal angles.*



Given $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$.

To prove $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \times AC}{A'B' \times A'C'}.$$

Proof. Draw the altitudes BD and $B'D'$.

$$\angle A = \angle A'. \qquad \text{(Hyp.)}$$

$$\therefore \triangle ABD \sim \triangle A'B'D'. \qquad \text{(Why ?)}$$

But $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AC \times BD}{A'C' \times B'D'}, \qquad \text{(Why ?)}$$

$$= \frac{AC}{A'C'} \times \frac{BD}{B'D'}, \qquad \text{(Why ?)}$$

$$= \frac{AC}{A'C'} \times \frac{AB}{A'B'}, \qquad \text{(Why ?)}$$

$$= \frac{AC \times AB}{A'C' \times A'B'}. \qquad \text{(Why ?)}$$

Q. E. D.

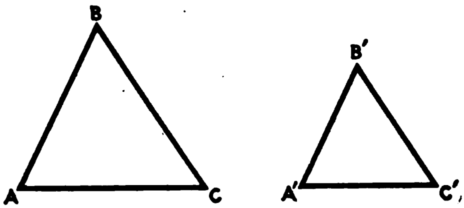
Ex. 1156. In triangles ABC and $A'B'C'$, $\angle A = \angle A'$, $AB = 6$ in., $AC = 9$ in., $A'B' = 1$ ft., and $A'C' = 2$ ft. Find the ratio of $\triangle ABC$ to $\triangle A'B'C'$.

Ex. 1157. Triangle ABC is equivalent to triangle $A'B'C'$, and $\angle A = \angle A'$. Find AC if $AB = 2$ in., $A'B' = 3$ in., and $A'C' = 4$ in.

Ex. 1158. If $\triangle ABC = \triangle A'B'C'$, and $\angle A = \angle A'$, then
$$AB : A'B' = A'C' : AC.$$

Proposition XIV. Theorem

379. *Similar triangles are to each other as the squares of their homologous sides.*



Given $\triangle ABC \sim \triangle A'B'C$

To prove $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}.$

Proof. $\angle A = \angle A'.$ (303)

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \times AC}{A'B' \times A'C'}, \qquad (378)$$

$$= \frac{AB}{A'B'} \times \frac{AC}{A'C'},$$

$$= \frac{AB}{A'B'} \times \frac{AB}{A'B'},$$

$$= \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

In like manner

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$

Q.E.D.

Ex. 1159. The sides of a triangle are 4, 7, and 8. What are the sides of a similar triangle whose area is four times as large ?

Ex. 1160. Similar triangles are to each other as the squares of homologous altitudes.

Ex. 1161. Find the ratio of two similar triangles, two of whose homologous sides are 2 and 3 in. respectively.

Ex. 1162. Construct a triangle nine times as large as a given triangle and similar to it.

Ex. 1163. The areas of two similar triangles are to each other as 4 : 5. Find the ratio of their homologous sides.

Ex. 1164. If in the diagram for Prop. XV,

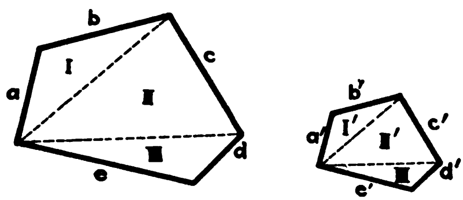
$$\text{polygon } abcde \sim \text{polygon } a'b'c'd'e',$$

and $a = 3\,a'$, find (*a*) the ratio of $\triangle I$ to $\triangle I'$, (*b*) the ratio $\triangle II$ to $\triangle II'$, (*c*) the ratio of polygon $abcde$ to polygon $a'b'c'd'e'$.

Ex. 1165. If in the same diagram $a = na'$, find (*a*) the ratio of $\triangle I$ to $\triangle I'$, (*b*) the ratio of $\triangle II$, to $\triangle II'$, (*c*) the ratio of polygon $abcde$ to polygon $a'b'c'd'e'$.

Proposition XV. Theorem

380. *Similar polygons are to each other as the squares of their homologous sides.*



Given a and a' homologous sides of two similar polygons whose areas are S and S' respectively.

To prove $\dfrac{S}{S'} = \dfrac{a^2}{a'^2}$.

Proof. From any two homologous vertices draw all possible diagonals.

Then, $\triangle I \sim \triangle I',\ \triangle II \sim \triangle II'$, etc. (Why ?)

Whence $\dfrac{\triangle I}{\triangle I'} = \dfrac{a^2}{a'^2}$. (Why ?)

$\dfrac{\triangle II}{\triangle II'} = \dfrac{b^2}{b'^2} = \dfrac{a^2}{a'^2}$. (Why ?)

$$\frac{\triangle III}{\triangle III'} = \frac{c^2}{c'^2} = \frac{a^2}{a'^2}.$$

$$\therefore \frac{\triangle I}{\triangle I'} = \frac{\triangle II}{\triangle II'} = \frac{\triangle III}{\triangle III'} = \frac{a^2}{a'^2}. \qquad \text{(Ax. 1.)}$$

$$\therefore \frac{\triangle I + \triangle II + \triangle III}{\triangle I' + \triangle II' + \triangle III'} = \frac{a^2}{a'^2}, \qquad (286)$$

or

$$\frac{S}{S'} = \frac{a^2}{a'^2}.$$

Q. E. D.

Ex. 1166. The areas of two similar polygons are respectively 3 sq. in. and 12 sq. in., and a side of one is 6 in. Find the homologous side of the other.

Ex. 1167. Two homologous sides of two similar polygons are 4 in. and 8 in. respectively, the area of the greater polygon is 1 sq. ft. What is the area of the smaller polygon ?

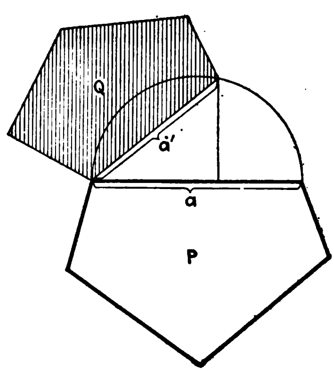
Ex. 1168. The area of a polygon is twice the area of a similar polygon. Find the ratio of any two homologous sides.

Ex. 1169. On a map whose scale is 1 : 1000, how many square feet are represented by a polygon equivalent to 2 sq. in. ?

Ex. 1170. Homologous sides of 2 similar polygons have the ratio of 5 to 9, the sum of their areas is 212 sq. ft. Find the area of each figure.

Ex. 1171. Two similar parallelograms are to each other as the products of their diagonals.

381. METHOD XXI. **As similar polygons (including triangles) are to each other as the squares of their homologous sides, Prop. IX may be used to draw a polygon that shall be any given part of a given polygon, and similar to it.**



Thus, the annexed diagram shows the construction of a polygon Q, which is equivalent to three sevenths of another polygon P, and similar to it. [That part of the construction which shows how to make Q similar to P is omitted (315).] Since

$$\frac{Q}{P} = \frac{a'^2}{a^2}, \text{ and } \frac{a'^2}{a^2} = \frac{3}{7}$$

(Prop. IX), the proof is obvious.

Q

Ex. 1172. Construct a triangle equivalent to two sevenths of a given triangle, and similar to it.

Ex. 1173. Construct a triangle similar to a given triangle and three times as large.

Ex. 1174. To divide a triangle into two equivalent parts by a line parallel to one of its sides.

Ex. 1175. To divide a triangle into five equivalent parts by lines parallel to one of its sides.

Ex. 1176. Construct a polygon equivalent to three fifths of a given polygon and similar to it.

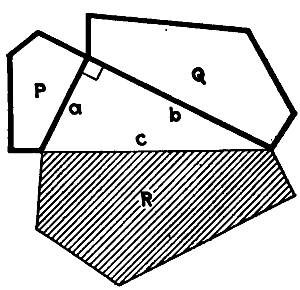
Ex. 1177. Construct a pentagon equivalent to three times a given pentagon and similar to it.

Ex. 1178. To divide a regular (*i.e.* equiangular and equilateral) pentagon into two equivalent parts so that one part is again a regular pentagon.

*** Ex. 1179.** To construct two lines having the same ratio as the areas of two similar given triangles.

Proposition XVI. Theorem

382. *If similar polygons are constructed on the three sides of a right triangle, the sum of the polygons on the arms is equivalent to the polygon on the hypotenuse.*



Given the similar polygons P, Q, and R constructed, respectively, on the arms a and b and the hypotenuse c of the rt. $\triangle abc$.

To prove $P + Q = R.$

Proof.
$$\frac{P}{R} = \frac{a^2}{c^2}. \qquad (380)$$

$$\frac{Q}{R} = \frac{b^2}{c^2}. \qquad (380)$$

$$\therefore \frac{P + Q}{R} = \frac{a^2 + b^2}{c^2} = 1. \qquad \text{(Ax. 2, 326.)}$$

I.e. $P + Q = R.$ Q. E. D.

383. Cor. If similar polygons are constructed on the three sides of a right triangle, the polygon on one arm is equivalent to the difference of the polygons on the hypotenuse and the other arm.

Ex. 1180. Construct an equilateral triangle equivalent to the sum of two given equilateral triangles.

Ex. 1181. Construct an equilateral triangle equivalent to the difference of two given equilateral triangles.

Ex. 1182. Construct an isosceles right triangle equivalent to the sum of two given isosceles right triangles.

Ex. 1183. To construct an isosceles right triangle equivalent to the difference of two isosceles right triangles.

Ex. 1184. Construct a triangle similar to two given similar triangles and equivalent to their sum.

Ex. 1185. To construct a polygon similar to two given similar polygons and equivalent to their sum.

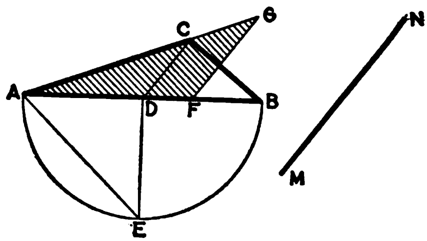
Ex. 1186. To construct a polygon similar to two given similar polygons and equivalent to their difference.

Ex. 1187. To construct an equilateral triangle equivalent to the sum of three given equilateral triangles.

Ex. 1188. Two homologous sides of two similar polygons are 12 feet and 5 feet respectively. Find the homologous side of a similar polygon equivalent to their sum.

PROPOSITION XVII. PROBLEM

384. *To transform the triangle ABC so that the angle A is not altered, and the side opposite the angle A becomes parallel to a given line MN.*



Construction. Draw $CD \parallel MN$, intersecting AB in D.
Construct AE, a mean proportional between AD and AB.
On AC lay off $AF = AE$.
Draw $FG \parallel MN$, meeting AC produced in G.
AFG is the required triangle.

Proof.

$$\frac{\triangle ABC}{\triangle AGF} = \frac{AB \times AC}{AF \times AG} = \frac{AB}{AF} \times \frac{AC}{AG} = \frac{AB}{AF} \times \frac{AD}{AF} = \frac{AB \times AD}{\overline{AF}^2} = 1.$$

$\therefore$ area ABC = area AGF. Q. E. F.

385. METHOD XXII. **While the fundamental method for transforming triangles (363) changes all angles of a triangle, Prop. IX enables us to transform a triangle without changing one angle.**

Ex. 1189. Transform $\triangle ABC$ into a right triangle so that angle A is not altered.

Ex. 1190. Transform $\triangle ABC$ so that angle A is not altered and another angle becomes equal to a given angle Q.

Ex. 1191. Transform $\triangle ABC$ into an isosceles triangle, having a base angle equal to angle A.

Ex. 1192. Transform $\triangle ABC$ into an isosceles triangle having the vertex angle equal to angle A.

Ex. 1193. Transform $\triangle ABC$ so that angle A is not altered and the including sides are to each other as 3 to 4.

Ex. 1194. Transform $\triangle ABC$ so that angle A is not altered and that another angle equals $\frac{C}{2}$.

Ex. 1195. Transform a given triangle into a right triangle containing a given acute angle.

HINT. Apply successively Methods XX and XXII.

Ex. 1196. To transform a given triangle into an isosceles triangle, having a given vertex angle.

Ex. 1197. To transform a given triangle into an equilateral triangle.

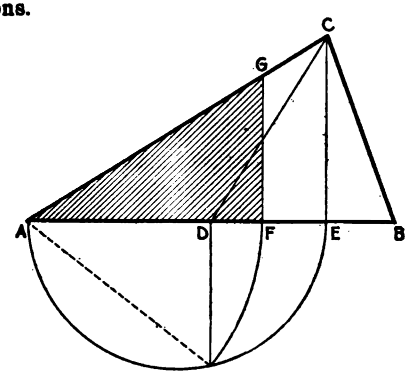
Ex. 1198. To transform a given triangle into another one containing two given angles.

Ex. 1199. To transform a given triangle into another one which contains a given angle, so that the sides including the angle shall be as 3 : 4.

Ex. 1200. To transform a given parallelogram into an equilateral triangle.

386. METHOD XXIII. **To construct a figure which shall be a given part $\left(\frac{m}{n}\right)$ of a given figure and which shall satisfy other conditions, first obtain the required part $\left(\frac{m}{n}\right)$ of the given figure by any method, and then transform the area thus obtained so as to satisfy the given conditions.**

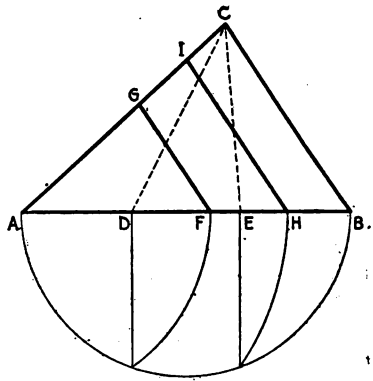
To obtain $\frac{m}{n}$ of a given polygon, we usually transform it into a triangle.



Ex. 1201. To cut off $\frac{1}{2}$ of a given $\triangle ABC$ by a line perpendicular to the base, draw the median CD, then

$$\triangle ADC = \tfrac{1}{2} \triangle ABC.$$

Transform $\triangle ADC$ so that angle A is not altered and the line opposite A becomes parallel to the altitude CE (Method XXII). $\triangle AFG$ is the required one.

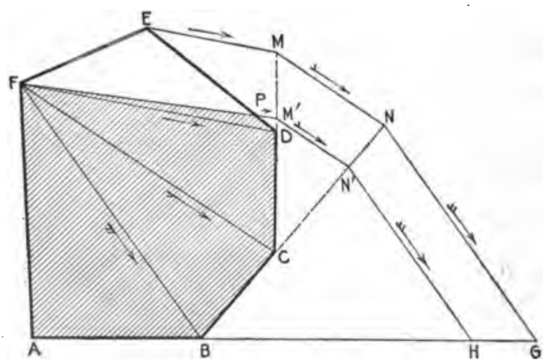


Ex. 1202. To divide $\triangle ABC$ into three equivalent parts by lines parallel to BC.

Trisect the base, then $\triangle ADC = \frac{1}{3}$, and $\triangle AEC = \frac{2}{3}$ of $\triangle ABC$. Hence transform these triangles so that $\angle A$ is not altered and the opposite sides become parallel to AC (Prop. XVII). Thus we obtain $\triangle$s AFG and AHI. Hence FG and HI are the required lines.

Ex. 1203. To cut off $\frac{9}{10}$ of a given polygon $ABCDEF$ by a line drawn from F, transform the polygon into $\triangle AGF$ (365), and make $AH = \frac{9}{10}\, AG$. Then $\triangle AHF = \frac{9}{10}$ of $ABCDEFG$.

To transform $\triangle AHF$ into a portion of the given polygon, make



$HN' \parallel GN$, $N'M' \parallel NM$, and $M'P \parallel ME$. Then $ABCDPF = \triangle AFH$ by (365). Hence FP is the required line.

387. Note. The preceding exercise leads easily to the solution of the problem of subdividing a polygon into n equivalent parts, and since all rectilinear areas are polygons, a great many problems are special cases of Ex. 1203.

Ex. 1204. To cut off $\frac{1}{3}$ of a triangle by a line parallel to a median.

Ex. 1205. To cut off $\frac{1}{3}$ of a triangle by a line parallel to an angle bisector.

Ex. 1206. To divide a triangle into three equivalent parts by lines perpendicular to the base.

Ex. 1207. To cut off $\frac{3}{5}$ of a given triangle by a line parallel to a given line.

Ex. 1208. To divide a quadrilateral into two equivalent parts.

Ex. 1209. To divide a quadrilateral into three equivalent parts.

Ex. 1210. To divide a triangle ABC into four equivalent parts by lines parallel to BC.

Ex. 1211. To bisect the area of a triangle by a line drawn from a point P in AC. (HINT. Consider the figure a quadrilateral, $ABCP$.)

Ex. 1212. Construct an equilateral triangle equivalent to one half of a given square.

Ex. 1213. To divide a given pentagon into three equivalent parts.

Ex. 1214. To construct a square equivalent to $\frac{3}{5}$ of a given triangle.

MISCELLANEOUS EXERCISES

THEOREMS

Ex. 1215. If E is any point in the diagonal AC of the parallelogram $ABCD$, prove that $\triangle AEB = \triangle ADE$.

Ex. 1216. A straight line passing through the intersections of the diagonals of a parallelogram divides the figure into two equivalent parts.

Ex. 1217. The area of a circumscribed polygon is equivalent to one half the product of perimeter and radius.

Ex. 1218. If through any point in a diagonal of a parallelogram, parallels are drawn to the sides, four parallelograms are formed, of which the two which do not contain the diagonal are equivalent.

Ex. 1219. If any point within a parallelogram be joined to the four vertices, the sum of either pair of opposite triangles is equivalent to one half the parallelogram.

Ex. 1220. The lines joining the mid-points of the sides of a quadrilateral in succession form a parallelogram equivalent to one half the quadrilateral.

Ex. 1221. The areas of two similar triangles are to each other as the squares of any two homologous angle bisectors.

Ex. 1222. The areas of two similar triangles are to each other as the squares of any two homologous medians.

Ex. 1223. If the median AX, BY, and CZ of $\triangle ABC$, meet in O, then $\triangle AOC$ is equivalent to $\triangle BOC$.

Ex. 1224. If on the line joining the mid-points of two sides of a triangle a parallelogram is constructed having two of its vertices in the base of the triangle, the parallelogram is equivalent to $\frac{1}{2}$ of the triangles.

Ex. 1225. The non-parallel sides of a trapezoid form with the diagonals two equivalent triangles.

Ex. 1226. If from the point of intersection of the medians of any triangle, lines are drawn to the three vertices, they form with the sides three equivalent triangles.

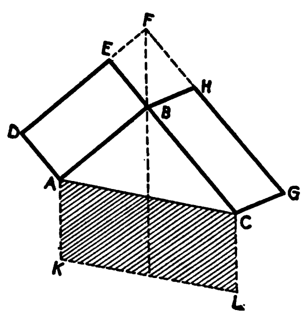
Ex. 1227. If in the triangle ABC, D and F are the mid-points of the sides AB and AC respectively, the triangles ADC and ABF are equivalent.

Ex. 1228. The area of an equilateral triangle whose side equals a is $\frac{a^2}{4}\sqrt{3}$.

Ex. 1229. The area of a rhombus is equal to half the product of the diagonals.

Ex. 1230. If on the line joining the mid-points of the two non-parallel sides of a trapezoid, a triangle be constructed whose vertex lies in the upper base of the trapezoid, the triangle is equivalent to $\frac{1}{2}$ the trapezoid.

Ex. 1231. If, on two sides of triangle ABC, parallelograms DB and BG are constructed, their sides DE and GH be produced to meet in F, and on AC a parallelogram be constructed, having AK equal and parallel to FB, then the parallelogram AL is equivalent to parallelogram AE plus parallelogram BG. (Pappus' Theorem.)



Ex. 1232. Find a similar proposition for triangles constructed on the three sides of a given triangle.

* **Ex. 1233.** A quadrilateral is equivalent to a triangle if its diagonals and the angle included between them are respectively equal to two sides and the included angle of the triangle.

* **Ex. 1234.** Two quadrilaterals are equivalent if the diagonals and the included angle of one are equal, respectively, to the diagonals and the included angle of the other. (Ex. 1233.)

* **Ex. 1235.** If two chords intersect within a circle at right angles, the sum of the squares upon their segments is equal to the square of the diameter.

PROBLEMS OF COMPUTATION

Ex. 1236. The side of an equilateral triangle is 10 in. Find the area.

Ex. 1237. Find the area of an isosceles triangle if the base is 6 and the arm 5.

Ex. 1238. Find the area of a trapezoid whose bases are 9 and 11 respectively, and whose altitude is 12 ft.

Ex. 1239. Find the area of a rhombus whose diagonals are 9 and 10 ft. respectively.

Ex. 1240. Find the area of quadrilateral $ABCD$ if $AB = 10$, $BC = 24$, $CD = 30$, $AD = 28$, diagonal $AC = 26$.

Ex. 1241. A side of equilateral triangle ABC is 8. Find the side of an equilateral triangle equivalent to three times triangle ABC.

Ex. 1242. The perimeter of a rectangle is 20 m., one side is 6 m. Find the area.

Ex. 1243. What is the side of a square whose area is 900 sq. m.? n sq. m.?

Ex. 1244. The area of a rhombus is equal to m, and one diagonal is equal to d. Find the other diagonal.

Ex. 1245. The area of a trapezoid is 400 sq. m., its altitude is 8 m. Find the length of the line joining the mid-points of the non-parallel sides.

Ex. 1246. The hypotenuse of a right triangle is 20, and the projection of one arm upon the hypotenuse is 4. What is its area?

Ex. 1247. A farmer wishes to determine the area of a pentagonal field. He measures the lines $AB = 4$ rods, $BC = 13$ rods, $CD = 14$ rods, $DE = 5$ rods, $EA = 12$ rods, $AC = 15$ rods, and $AD = 13$ rods. How many square rods does the field contain?

Ex. 1248. The base and altitude of a triangle are 12 and 20 respectively. At a distance of 6 from the base, a parallel is drawn to the base. Find the areas of the two parts of the triangle.

Ex. 1249. Find the area of a rectangle having one side equal to 6 and a diagonal equal to 10.

Ex. 1250. Find the area of a polygon whose perimeter equals 20 ft., circumscribed about a circle whose radius is 3 ft.

Ex. 1251. Find the area of a triangle whose base is 10 inches and whose base angles are $120°$ and $30°$ respectively.

Ex. 1252. Find the side of an equilateral triangle equivalent to a parallelogram, whose base and altitude are 10 and 15 respectively.

Ex. 1253. The chord of an arc is 42 in.; the chord of one half that arc is 29 in. Find the diameter of the circle.

Ex. 1254. The base of a triangle is 15 ft., its area is 60 sq. ft.; find the area of a similar triangle whose altitude is 6 ft.

Ex. 1255. An arm of an isosceles trapezoid is 13 in. and its projection on the longer base is 5 in.; the longer base is 17 in. Find the area of the trapezoid.

Ex. 1256. The arms of two isosceles right triangles are 3 and 2 respectively. Find the arm of an isosceles right triangle equivalent to their difference.

Ex. 1257. The sides of a triangle are as $8 : 15 : 17$. Find the sides if the area is 960 sq. ft.

Ex. 1258. The sides of a triangle are 8, 15, and 17. Find the radius of the inscribed circle.

Hint. Express in the form of an equation the fact that the area of the triangle is equal to the sum of the three triangles whose vertex is the incenter and whose bases are the sides of the triangle.

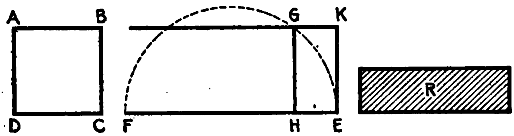
Ex. 1259. Find the area of an equilateral polygon of 12 sides inscribed in a circle whose radius is 4 in.

Ex. 1260. The sides of a triangle are 6, 7, and 8 ft. Find the areas of the two parts into which the triangle is divided by the bisector of the angle included by 6 and 7.

Ex. 1261. Find the area of an equilateral triangle whose altitude is equal to h.

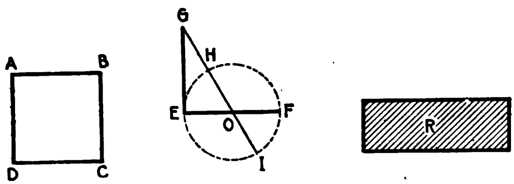
PROBLEMS OF CONSTRUCTION

Ex. 1262. To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.



Hint. If FE is the given line, and $ABCD$ the given square, make $EK = AB$.

Ex. 1263. To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.



Hint. If $ABCD$ is the given square and EF the given line, make $EG = AB$.

Ex. 1264. To transform a rectangle into another one, having given one side.

Ex. 1265. To transform a square into an isosceles triangle, having a given base.

Ex. 1266. To transform a rectangle into a parallelogram, having a given diagonal.

Ex. 1267. To divide a triangle into three equivalent parts by lines parallel to a median.

Ex. 1268. To bisect a parallelogram by a line perpendicular to a side.

Ex. 1269. To divide a parallelogram into three equivalent parts by lines drawn through a vertex.

Ex. 1270. To bisect a trapezoid by a line drawn through a vertex.

Ex. 1271. Divide a pentagon into four equal parts by lines drawn through one of its vertices.

Ex. 1272. Divide a quadrilateral into four equal parts by lines drawn from a point in one of its sides. (Consider the figure a pentagon.)

Ex. 1273. Find a point within a triangle such that the lines joining the point to the vertices shall divide the triangle into three equivalent parts.

Ex. 1274. Construct a square that shall be four fifths of a given triangle.

Ex. 1275. Construct an equilateral triangle that shall be two thirds of a given rectangle.

Ex. 1276. Find a point within a triangle such that the lines joining the point with the vertices shall form three triangles, having the ratio $3:4:5$.

Ex. 1277. Divide a given line into two segments such that one segment is to the line as $\sqrt{2}$ is to $\sqrt{5}$.

Ex. 1278. To transform a triangle into a right isosceles triangle.

Ex. 1279. Construct a triangle similar to a given triangle and equivalent to another given triangle.

Ex. 1280. To divide a triangle into three equivalent parts by lines parallel to a given line.

* **Ex. 1281.** Bisect a trapezoid by a line parallel to the bases.

[For practical applications see p. 300.]

BOOK V

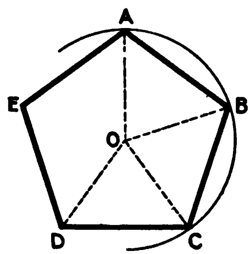
REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE

REGULAR POLYGONS

PROPOSITION I. THEOREM

388. DEF. A **regular polygon** is a polygon which is both equiangular and equilateral.

389. *A circle can be circumscribed about any regular polygon.*



Given $ABCDE$, a regular polygon of n sides.

To prove that a circle can be circumscribed about $ABCDE$.

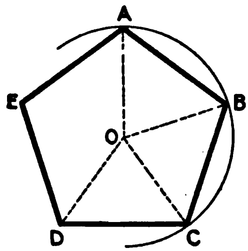
Proof. Construct a circle through A, B, and C, and let O be its center.

Draw OA, OB, OC, and OD.

$$\angle ABC = \angle BCD. \qquad \text{(Why ?)}$$

$$\angle OBC = \angle OCB. \qquad \text{(Why ?)}$$

$$\therefore \angle OBA = \angle OCD.$$



But $OC = OB$ and $AB = CD$. (Why?)

$\triangle OBA \cong \triangle OCD$. (Why?)

$\therefore OD = OA$.

$\therefore$ the circle passes through D.

In like manner, it may be proved that the circle passes through the remaining vertices of the polygon.

$\therefore$ a circle can be circumscribed about the given polygon.

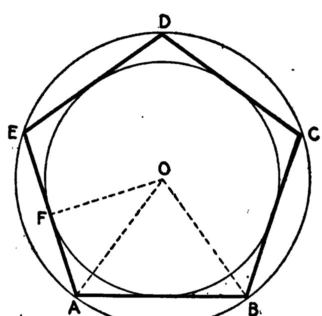
Q. E. D.

Proposition II. Theorem

390. *A circle can be inscribed in any regular polygon.*

Hint. Circumscribe a circle about the given polygon and prove that the center is equidistant from the sides. (197)

391. Def. The **center** (O) **of a regular polygon** is the common center of the circumscribed and inscribed circles of the polygon.



392. Def. The **radius of a regular polygon** is the radius of the circumscribed circle; as OA.

393. Def. The **central angle** is the angle between two radii drawn to the ends of one side; as AOB.

394. DEF. The **apothem of the polygon** is the radius of the inscribed circle; as OF.

395. COR. The central angle of a regular polygon of n sides is equal to $\frac{4}{n}$ right angles.

Ex. 1282. If $ABCDEF$ is a regular hexagon, then
$$\angle ADC = \angle AEC = \angle AFC.$$

Ex. 1283. If two diagonals AC and BE of a regular polygon $ABCDEFG$ intersect in Q, then $AQ \times QC = BQ \times QE$.

Ex. 1284. If in the regular heptagon $ABCDEFG$, the prolongations of AB and the diagonal of EC meet in H, then $HB \times HA = HC \times HE$.

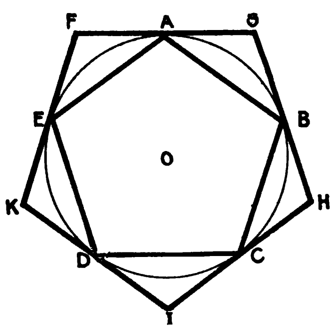
Ex. 1285. Any central angle of a regular polygon is the supplement of an angle of the polygon.

Ex. 1286. A triangle is regular if the centers of the circumscribed and inscribed circles coincide.

Ex. 1287. A polygon is regular if the centers of the circumscribed and inscribed circles coincide.

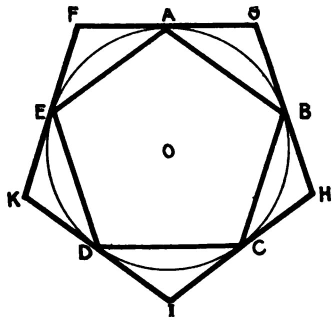
PROPOSITION III. THEOREM

396. *If the circumference of a circle is divided into any number of equal parts:*



(1) *The chords joining the points of division successively form a regular inscribed polygon.*

(2) *Tangents drawn at the points of division form a regular circumscribed polygon.*



Given the circumference ACE divided into the equal arcs AB, BC, CD, etc.

(1) **To prove** $ABCDE$ is a regular polygon.

Proof. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$, etc. (Why ?)

$\therefore \overset{\frown}{AC} = \overset{\frown}{BD} = \overset{\frown}{CE}$, etc. (Ax. 2.)

$\therefore \angle ABC = \angle BCD$, etc. (Why ?)

But $AB = BC = CD$. (Why ?)

$\therefore$ polygon $ABCDE$ is regular. (Why ?)

(2) **To prove** tangents drawn at A, B, C, etc., form the regular circumscribed polygon $FGHIK$.

Proof. $\angle GAB = \angle GBA = \angle CBH = \angle HCB$, etc., (Why ?)

and $AB = BC = CD$, etc. (Why ?)

$\therefore$ $\triangle$ ABG, CBH, CDI, etc., are congruent and isosceles. (Why ?)

$\therefore \angle G = \angle H = \angle I$, etc.

and $AG = GB = BH = HC$, etc.

Whence $GH = HI = IK$, etc. (Ax. 2.)

$\therefore$ circumscribed polygon $FGHIK$ is regular. Q. E. D.

397. Cor. 1. The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides.

398. Cor. 2. The perimeter of a regular circumscribed polygon is greater than the perimeter of a regular circumscribed polygon of double the number of sides.

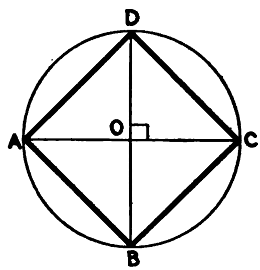
Ex. 1288. If a regular hexagon is inscribed in a circle, and the midpoints of the arcs which are subtended by the sides are joined to the vertices of the hexagon, a regular polygon of twelve sides is formed.

Ex. 1289. An equilateral polygon inscribed in a circle is regular.

Ex. 1290. An equiangular polygon circumscribed about a circle is regular.

Proposition IV. Problem

399. *To inscribe a square in a given circle.*



Construction. In the given circle ABC, draw diameters AC and BD perpendicular to each other, and draw AB, BC, CD, and DA.

Then $ABCD$ is the required square.

[The proof is left to the student.]

400. Cor. 1. By bisecting the central angles, the arcs AB, BC, etc., will be bisected, and a polygon of eight sides may be inscribed in the circle. By repeating the process, polygons of 16, 32, $\cdots$, 2^n sides may be constructed.

R

401. Cor. 2. By drawing tangents at A, B, C, and D, a square may be circumscribed about the circle.

402. Cor. 3. If R is the radius of a circle, the side of the inscribed square equals $R\sqrt{2}$.

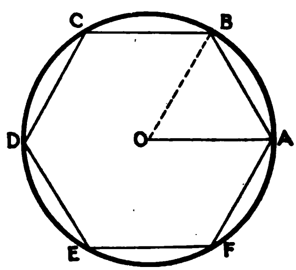
Ex. 1291. To circumscribe a regular octagon about a given circle.

Ex. 1292. To construct a regular octagon, having a given side.

Ex. 1293. Find the area of a square, if its radius is equal to r.

Proposition V. Problem

403. *To inscribe a regular hexagon in a given circle.*



Construction. In the given circle ACD, draw the radius AO.

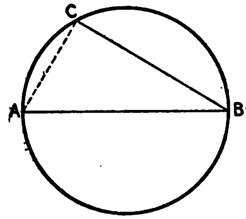
From A as a center, with a radius equal to OA, draw an arc meeting the circle in B. Draw AB.

[To be completed by the student.]

404. Cor. 1. By joining the alternate vertices of an inscribed regular hexagon, an inscribed equilateral triangle is formed.

405. Cor. 2. Regular polygons of 3, 6, 12, 24, etc., sides may be inscribed in and circumscribed about a given circle.

406. Cor. 3. If R is the radius of a circle, the side of the inscribed equilateral triangle is $R\sqrt{3}$.



Hint. If AB is a diameter, and $AC = R$, then BC is the required side.

Ex. 1294. If the radius of a circle is r, the apothem of the inscribed hexagon is equal to $\frac{r}{2}\sqrt{3}$.

Ex. 1295. To circumscribe a regular hexagon about a given circle.

Ex. 1296. To construct a regular polygon of twelve sides, having given a side.

Ex. 1297. Find the area of a regular hexagon if its radius is equal to r.

Ex. 1298. The apothem of an equilateral triangle is equal to one half its radius.

Ex. 1299. The area of an inscribed equilateral triangle is equal to one half the area of the inscribed regular hexagon.

Ex. 1300. The areas of triangles inscribed in equal circles are to each other as the products of their three sides. (339.)

Ex. 1301. A square constructed on a diameter of a circle is equivalent to twice the area of the inscribed square.

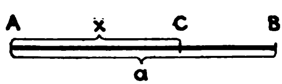
407. Def. A straight line is said to be divided in **extreme and mean ratio** when it is divided into two segments, such that *the greater is the mean proportional between the smaller and the whole line.*

Thus, AB is divided by C in extreme and mean ratio, if

$$AB : AC = AC : CB.$$

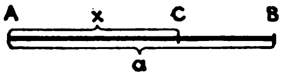
Proposition VI. Problem

408. *To divide a line in extreme and mean ratio.*



Given line $AB = a$.

Required to divide a in extreme and mean ratio.



Analysis. Suppose AC, or x, was the greater of the required segments, then $$CB = a - x.$$

Hence, $$a : x = x : a - x.$$

$$\therefore x^2 = a^2 - ax.$$

Transposing, $$x^2 + ax = a^2.$$

Completing the square,

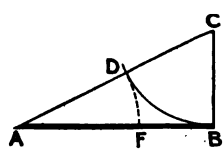
$$x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2.$$

Extracting the root, $$x + \frac{a}{2} = \sqrt{a^2 + \left(\frac{a}{2}\right)^2}.$$

$$\therefore x = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2}.$$

But since $\sqrt{a^2 + \left(\frac{a}{2}\right)^2}$ is the hypotenuse of a right triangle whose arms are a and $\frac{a}{2}$, x is easily constructed.

Construction. At B draw $CB = \frac{a}{2}$ and $\perp AB$.



$\left[\text{Then } AC = \sqrt{a^2 + \left(\frac{a}{2}\right)^2}.\right]$

On CA lay off $CD = CB\left(= \frac{a}{2}\right)$.

$\left[\text{Then } AD = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2} \text{ or } x.\right]$

On AB lay off $AF = AD$.

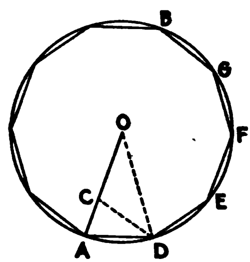
Then AB is divided as required. Q. E. F.

The proof of this construction is contained in the analysis. If we wish to write it out in detail, we may copy the last six lines of the analysis *in reverse order,* and assign reasons for each step.

409. NOTE. The method used for the discovery of the preceding problem is based upon algebra. Hence, it is usually called *algebraic analysis.* For further details of this method see appendix.

PROPOSITION VII. PROBLEM

410. *To construct the side of a regular decagon inscribed in a given circle.*



Given $\odot O$.

Required to construct the side of a regular inscribed decagon.

Construction. Divide a radius OA in extreme and mean ratio.

Draw chord AD equal to OC, the greater segment of OA.

AD is the side of the required decagon.

Proof. Draw DO and DC.

$$AO : OC = OC : CA, \qquad \text{(Con.)}$$

or since $$OC = AD,$$

$$AO : AD = AD : AC. \qquad \text{(Sub.)}$$

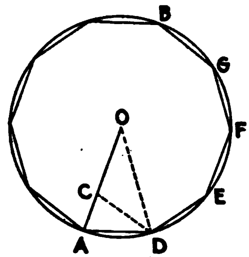
$$\therefore \triangle OAD \sim \triangle CAD. \qquad (309)$$

$$\therefore \angle O = \angle CDA. \qquad (303)$$

But $\triangle OAD$ being isosceles, the similar triangle DAC must be isosceles,

or, $$CD = AD = CO,$$

whence $$\angle O = \angle DCO. \qquad \text{(Why ?)}$$



But $\angle O = \angle CDA$.

$\therefore 2\angle O = \angle D$, (Ax. 2.)

and $2\angle O = \angle A$.

But $\angle O + \angle D + \angle A = 2$ rt. $\angle$s.

$\therefore 5\angle O = 2$ rt. $\angle$s,

or $\angle O = \frac{2}{5}$ rt. $\angle$, or $\frac{1}{10}$ of 4 rt. $\angle$s.

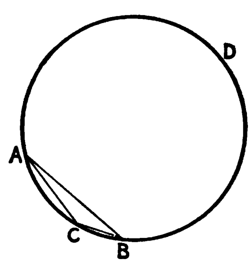
$\therefore$ arc AD is $\frac{1}{10}$ of the circumference, and AD is the side of an inscribed regular decagon. Q. E. F.

411. Cor. 1. By joining the alternate vertices A, E, G, etc., an inscribed regular pentagon is formed.

412. Cor. 2. Regular polygons of 5, 10, 20, etc., sides may be inscribed in and circumscribed about a given circle.

Proposition VIII. Problem

413. *To construct the side of a regular polygon of fifteen sides inscribed in a given circle.*



Given $\odot O$.

Required to construct the side of an inscribed regular polygon of fifteen sides.

Construction. In the given circle ABD, draw the chord AB equal to the radius, and chord AC equal to the side of the inscribed regular decagon.

Then CB is the side of the inscribed regular polygon of fifteen sides.

Proof. Arc $AB = \frac{1}{6}$ of the circumference.

Arc $AC = \frac{1}{10}$ of the circumference.

Arc $CB = \frac{1}{6} - \frac{1}{10}$ or $\frac{1}{15}$ of the circumference,

and CB is the side of the required polygon. Q. E. D.

414. Cor. Polygons of 15, 30, 60, etc., sides may be inscribed in and circumscribed about a given circle.

Ex. 1302. The side of what kind of polygon is CB (diagram of Prop. VIII), if (*a*) AB is the side of a regular pentagon, and AC the side of a regular hexagon? (*b*) AB is the diagonal of a regular pentagon, and AC is the side of an equilateral triangle?

Ex. 1303. If R is the radius of a circle, the side of the inscribed regular decagon equals $\frac{R}{2}(\sqrt{5}-1)$.

Hint. Simplify the value of x found in the analysis of Prop. VI. $(a=R.)$

Ex. 1304. To construct an angle of 36°.

Ex. 1305. To construct a regular decagon, having given a side.

Ex. 1306. To divide a right angle into five equal parts.

Ex. 1307. The diagonals of a regular pentagon are equal.

Ex. 1308. The diagonals of a regular pentagon divide each other in extreme and mean ratio.

Ex. 1309. The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Ex. 1310. Any radius of a regular polygon bisects an angle of the polygon.

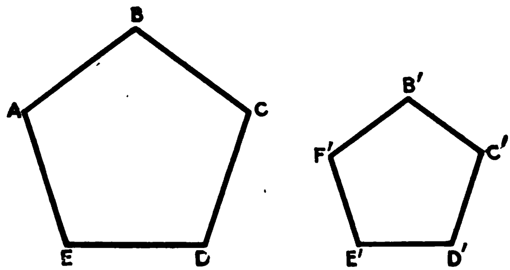
Ex. 1311. The diagonals drawn from a vertex of a regular decagon divide that angle into eight equal parts.

Ex. 1312. To construct a regular pentagon, having given a side.

[For practical applications see problems 91–100, p. 301.]

Proposition IX. Theorem

415. *Regular polygons of the same number of sides are similar.*



Given $ABCDE$ and $A'B'C'D'E'$ regular polygons of n sides.

To prove $ABCDE \sim A'B'C'D'E'$.

Proof. Each angle of both polygons is equal to $\frac{n-2}{n}$ st. $\angle s$. (Why?)

$\therefore$ polygons $ABCDE$ and $A'B'C'D'E'$ are mutually equiangular.

Since $AB = BC$, etc., and $A'B' = B'C'$, etc., (Why?)

$$AB : A'B' = BC : B'C' = \text{etc.}$$

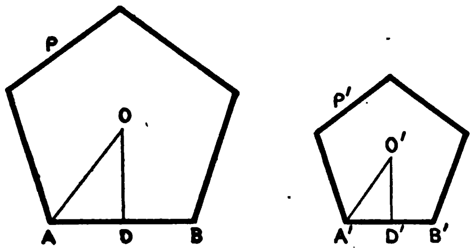
$\therefore$ $ABCDE$ and $A'B'C'D'E'$ are similar. Q. E. D.

Ex. 1313. To construct a regular hexagon equivalent to one half of a given regular hexagon.

Ex. 1314. Construct a regular pentagon equivalent to the sum of two given regular pentagons.

PROPOSITION X. THEOREM

416. *The perimeters of regular polygons of the same number of sides have the same ratio as their radii or as their apothems.*



Given P and P' the perimeters of two regular polygons of the same number of sides, having the radii OA and $O'A'$, and the apothems OD and $O'D'$ respectively.

To prove $P : P' = OA : O'A' = OD : O'D'$.

Proof $\angle AOD = \angle A'O'D'$. (Ax. 8.)

$\therefore \triangle OAD \sim \triangle O'A'D'$. (306)

Hence, $AD : A'D' = OD : O'D' = AO : A'O'$. (303)

But $P : P' = AB : A'B' = AD : A'D'$. (Why?)

$\therefore P : P' = OD : O'D' = AO : A'O'$. Q. E. D.

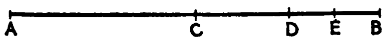
417. COR. The areas of regular polygons of the same number of sides are to each other as the squares of their radii or apothems.

Ex. 1315. The lines joining the mid-points of the radii of a regular pentagon inclose a regular pentagon equivalent to one fourth the given pentagon.

418. DEF. A **constant** is a quantity that maintains the same value throughout a given discussion. A **variable** is a quantity whose value changes during the same discussion.

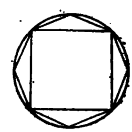
419. DEF. If a variable x approaches a constant a so that the difference between a and x becomes less than any conceivable number, then a is called the **limit** of x.

1. For example, suppose a point P to move from A toward B in such a way as to move in the first second over half of AB to C; in the second second over half of the remainder, CB to D; in the third second over half of the new remainder, DB to E; and so on indefinitely. It is evident that the distance from A to the moving point P is a variable, whose value can be made to differ from AB by less than any assigned quantity (although it never can be made equal to AB).



Hence, AB is the *limit of the variable AP.*

2. Inscribe a square in a circle, then inscribe an octagon by joining the mid-points of the four equal arcs to the vertices, and by continuing this process obtain regular inscribed polygons of 16, 32, 64, etc. sides. Obviously the area of the circle C is larger than the area of any of these polygons A, but the difference between these two areas, C-A, becomes less if we increase the number of sides of the polygon. By continually increasing the number of sides of the polygon, the area of the polygon will approach the area of the circle more and more, or C-A can be made less than any assignable quantity, however small.



Hence, the area of the circle is the limit of the area of the polygon.

3. Similarly the perimeter of an inscribed polygon increases with the number of sides (397) and approaches the circumference so that the difference between circumference and perimeter becomes less than any assignable number. Hence, the circumference is the limit of the increasing perimeter.

4. Consider the infinite periodic decimal .999 . . . If we take one decimal, the value differs from 1 by .1, for two decimals this difference is .01, for three decimals it is .001, and so forth. Hence the difference between 1 and the decimal can be made less than any assignable number however small. Or 1 is the limit of the decimal .999 ⋯.

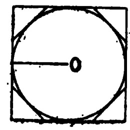
420. NOTE. In elementary geometry the variables that approach a limit as a rule cannot become equal to this limit. There are, however, cases in which variables become equal to their limits.

Ex. 1316. What is the limit of the periodic decimal .666 ⋯ ?

Ex. 1317. What is the limit of the fraction $\frac{1}{n}$, if n increases indefinitely?

Ex. 1318. What is the limit of the sum $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$?

Ex. 1319. What is the limit of the area of a regular circumscribed polygon of n sides if n assumes the values 4, 8, 16, ⋯ to infinity?

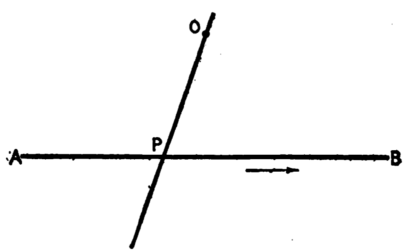


Ex. 1320. What is the limit of the perimeter of a regular circumscribed polygon of n sides, if n assumes the values 4, 8, 16, ⋯ to infinity?

Ex. 1321. If the number of sides of a regular inscribed polygon is increased indefinitely, what is the limit of (a) the apothem? (b) the side?

Ex. 1322. If the number of sides of a regular circumscribed polygon is increased indefinitely, what is the limit of its radius?

Ex. 1323. A point P moves on a fixed straight line AB, and another straight line OP always passes through a fixed point O and the moving point P. What is the limiting position of OP, if P moves to a distance greater than any conceivable number?



421. THEOREM. **If two variables, x and y, are always equal and x approaches a as a limit, then y approaches a as a limit.**

Since $a - x = a - y$, and $a - x$ can be made less than any conceivable number,

$a - y$ can be made less than any conceivable number.

$\therefore$ y approaches a as a limit.

422. COR. **If two variables are always equal, and each approaches a limit, their limits are equal.** (PRINCIPLE OF LIMITS.)

423. Def. **The length of a circumference** (or briefly a circumference) is the limit of the perimeter of a regular inscribed polygon the number of whose sides is increased indefinitely.

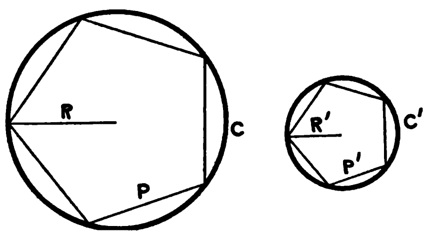
The study of the third illustration in (419) will explain to the student the true meaning of this definition.

Having defined a circumference, the meaning of the length of an arc is determined by § 228.

Note. It can be demonstrated that the limit of the perimeter of *circumscribed* regular polygons is the same as that of the inscribed ones. Hence a circumference can be defined as a common limit of inscribed and circumscribed regular polygons. For the purposes of this textbook, however, it is sufficient to consider the inscribed regular polygons only.

Proposition XI. Theorem

424. *Circumferences are to each other as their radii.*



Given C and C' any two circumferences with radii R and R' respectively.

To prove $C : C' = R : R'$.

Proof. Inscribe regular polygons of n sides in each circle having the perimeters P and P' respectively.

$$\frac{P}{P'} = \frac{R}{R'}, \qquad (416)$$

or

$$\frac{P}{R} = \frac{P'}{R'}. \qquad (281)$$

Increase n indefinitely, then the equal variables $\frac{P}{R}$ and $\frac{P'}{R'}$ must have equal limits.

But C and C' are the limits of the P and P' respectively. (423)

$$\therefore \frac{C}{R} = \frac{C'}{R'}, \qquad (422)$$

or $$C : C' = R : R'. \qquad \text{Q.E.D.}$$

425. COR. 1. Circumferences are to each other as their diameters.

426. COR. 2. The ratio of the circumference to its diameter is constant.

Since $$C : C' = 2\,R : 2\,R',$$

$$\frac{C}{2\,R} = \frac{C'}{2\,R'}.$$

That is, the ratio of any circumference to its diameter is equal to the ratio of any other circumference to *its* diameter.

$\therefore$ the ratio of the circumference to its diameter is constant.

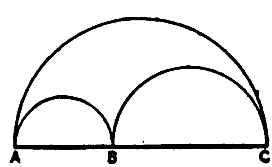
427. The constant value of $\frac{C}{2\,R}$ is denoted by the Greek letter π. It is an irrational number, which can only be found approximately, but with any degree of precision.

Since $$\frac{C}{2\,R} = \pi \quad C = 2\,\pi R.$$

Ex. 1324. Construct a circumference equal to the sum of two given circumferences.

Ex. 1325. Draw three circumferences which are to each other as $4:5:6$.

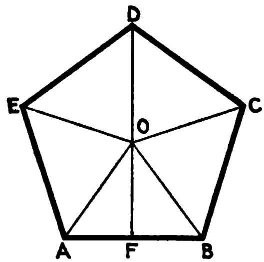
Ex. 1326. If B is any point in AC and on AB, BC and AC as diameters semicircles are drawn, prove that the largest semicircumference equals the sum of the two smaller ones.



[Practical applications on p. 302.]

PROPOSITION XII. THEOREM

428. *The area of a regular polygon is equal to one half the product of its apothem and perimeter.*



Given S, the area, P the perimeter, and r the apothem of the polygon $ABCDE$, having n sides.

To prove $S = \frac{1}{2} P \times r$.

Proof. Draw radii OA, OB, etc., dividing the polygon into n equal triangles.

$$\triangle ABO = \tfrac{1}{2} AB \times r.$$

$$S = n \times \triangle ABO = \tfrac{1}{2} n \times AB \times r.$$

But $n \times AB = P$,

or $S = \frac{1}{2} P \times r$. Q.E.D.

429. DEF. A **sector** of a circle is a figure bounded by two radii and their intercepted arc.

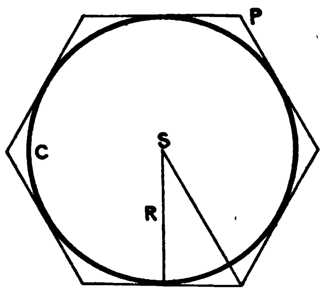
430. DEF. **Similar arcs, segments,** and **sectors** are those which correspond to equal central angles.

431. DEF. The **area of a circle** is the limit of the area of a regular circumscribed polygon the number of whose sides is increased indefinitely.

NOTE. It can be proved that the areas of regular inscribed polygons approach the same limit as the areas of the circumscribed polygons. For the purposes of this book, however, it is more convenient to consider the circumscribed polygon only.

PROPOSITION XIII. THEOREM

432. *The area of a circle is equal to one half the product of its circumference and radius.*



Given S the area of a circle of radius R and circumference C.

To prove $S = \frac{1}{2}\,C \times R$.

Proof. Circumscribe a regular polygon about the circle, and let P be its perimeter, A its area.

$$A = \tfrac{1}{2}\,P \times R. \qquad (428)$$

Let the number of sides be increased indefinitely, then

A approaches S as a limit. (423)

P approaches C as a limit. (423)

$$\therefore S = \tfrac{1}{2}\,C \times R. \qquad (422)$$

Q. E. D.

433. COR. 1. The area of a circle is equal to π times the square of its radius.

For $S = \frac{1}{2}\,C \times R$ and $C = 2\,\pi R$.

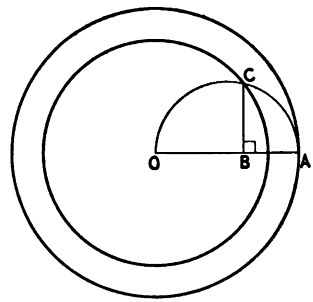
$$\therefore S = \pi R^2.$$

434. COR. 2. The areas of circles are to each other as the squares of their radii.

435. COR. 3. The area of a sector whose central angle is $n°$ is equal to $\dfrac{n}{360}\,\pi R^2$.

436. Cor. 4. Similar sectors are to each other as the squares of their radii.

437. Method XXIV. **Since the areas of circles have the same ratio as the squares of their radii, a great many constructions referring to squares may be applied to circles.**



Ex. 1327. To construct a circle equivalent to $\frac{3}{5}$ of a circle. The radius of the required circle is found by constructing a line whose square is $\frac{3}{5}$ of the square of the given radius (OA). Hence, make $OB = \frac{3}{5}\ OA$, and construct OC, the mean proportional between OA and OB. The circle constructed with OC as a radius is the required one.

Ex. 1328. Construct a circle equivalent to three sevenths of a given circle.

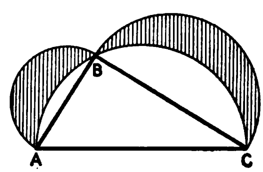
Ex. 1329. Construct a circle equivalent to three times a given circle.

Ex. 1330. Construct a semicircle equivalent to a given circle.

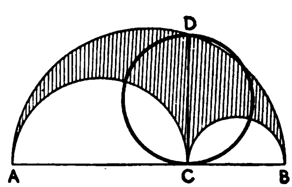
Ex. 1331. Construct a circle equivalent to the sum of two given circles.

Ex. 1332. Construct a circle equivalent to the difference of two given circles.

Ex. 1333. Construct a circle equivalent to the area bounded by two concentric circumferences.



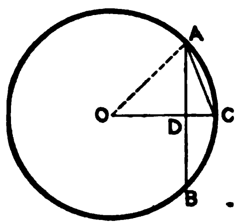
* **Ex. 1334.** If upon the three sides of a right triangle semicircles be drawn as indicated in the diagram, the area of the right triangle is equivalent to the sum of the two crescent-shaped areas, bounded by the semicircles. (Hippocrates' Theorem.)



* **Ex. 1335.** If C is a point in the straight line AB, the three semicircles, drawn respectively upon AB, AC, and CB as diameters, bound an area equivalent to a circle whose diameter is the perpendicular CD, D being in the largest semicircle.

PROPOSITION XIV. PROBLEM

438. *The side and the radius of a regular polygon being given, to compute the side of a regular polygon of the same radius and double the number of sides.*



Given AB equal to a side s of a regular polygon inscribed in $\odot O$ with radius $OA = R$.

Required. To compute AC, the side of a regular polygon of double the number of sides, inscribed in the same circle.

Construction. Draw CO meeting AB in D.

OC bisects AB at right angles. (80)

$$\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2\ OC \times OD, \qquad (331)$$

or $$AC^2 = 2\ R^2 - 2\ R \times OD. \qquad \text{(Sub.)}$$

But $$\overline{OD}^2 = R^2 - \left(\frac{s}{2}\right)^2. \qquad (327)$$

$$\therefore OD = \sqrt{R^2 - \frac{s^2}{4}},$$

or $$OD = \tfrac{1}{2}\sqrt{4\ R^2 - s^2}.$$

Hence $$\overline{AC}^2 = 2\ R^2 - R\sqrt{4\ R^2 - s^2}.$$

$$\therefore AC = \sqrt{2\ R^2 - R\sqrt{4\ R^2 - s^2}}.$$

439. COR. If $R = 1$, and the side of a polygon of n sides equals s_n, then

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

Ex. 1336. The side of an inscribed square is equal to $R\sqrt{2}$. Prove that the side of the regular octagon equals $\sqrt{2-\sqrt{2}}$.

Ex. 1337. If R is the radius of a regular polygon of twelve sides, its side equals $R\sqrt{2-\sqrt{3}}$.

Proposition XV. Problem

440. *To compute the ratio of the circumference of a circle to the diameter.*

Let $R = 1$, then $s_6 = 1$,
and according to (439), we have

	Length of Side	Length of Perimeter
$s_{12} = \sqrt{2-\sqrt{4-1^2}}$	$= 0.51764$	6.21166
$s_{24} = \sqrt{2-\sqrt{4-(0.51764)^2}}$	$= 0.26105$	6.26526
$s_{48} = \sqrt{2-\sqrt{4-(0.26105)^2}}$	$= 0.13081$	6.27870
$s_{96} = \sqrt{2-\sqrt{4-(0.13081)^2}}$	$= 0.06534$	6.28206
$s_{192} = \sqrt{2-\sqrt{4-(0.06534)^2}}$	$= 0.03272$	6.28291
$s_{384} = \sqrt{2-\sqrt{4-(0.03272)^2}}$	$= 0.01636$	6.28312
$s_{768} = \sqrt{2-\sqrt{4-(0.01636)^2}}$	$= 0.00818$	6.28317

Taking the last perimeter as the approximate value of the circumference,

$$\pi = \frac{C}{2R} = \frac{6.28317}{2} = 3.14159.$$

441. Method XXV **Most computations referring to the circle involve the use of the two formulæ $C = 2\pi R$ and $S = \pi R^2$. If R is not given, compute R first.**

Ex. 1338. Find the circumference of a circle whose radius is 5.

Ex. 1339. The diameter of a circle equals 10 in. Find the length of an arc of (*a*) 60°, (*b*) 1°, (*c*) 5°.

Ex. 1340. Find the area of a circle whose radius is 10.

Ex. 1341. Find the radius of a circle whose area is 100 sq. in.

Ex. 1342. Find the radius of a circle whose circumference equals m inches.

Ex. 1343. Find the circumference of a circle whose area equals 100π.

Ex. 1344. Find the area of a circle whose circumference equals 4.

Ex. 1345. Find the circumference of a circle whose area equals S.

Ex. 1346. Find the area of a circle whose circumference equals C.

Ex. 1347. Find the radius of a circle equivalent to a square whose side equals 6.

Ex. 1348. The circumference of a circle equals 10. Find the circumference of a circle having twice the area of the given circle.

Ex. 1349. Two concentric circles have their circumferences equal to 30 and 40 respectively. Find the area bounded by the two circumferences.

Ex. 1350. Find a semicircle equivalent to an equilateral triangle whose side equals 5.

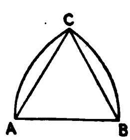
Ex. 1351. Find the area of a sector whose radius equals 5, and whose central angle equals 40°.

Ex. 1352. A square is inscribed in a circle of radius 10. Find the area of a segment bounded by a side of the square.

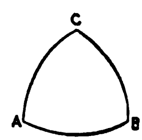
Ex. 1353. The radius of a circle is 4. What is the area of a segment whose arc is (a) 120°, (b) 60°?

Ex. 1354. In the diagram given here, $AB = BC = CA = 4$, B is the center of arc AC, and A is the center of arc BC. Find the total area.

Ex. 1355. A is the center of arc BC, B the center of AC, and C the center of AB. Find the total area, if $AB = 6$ in.



Ex. 1354



Ex. 1355

Ex. 1356. A circle has an area of 60 sq. ft. Find the length of an arc of 60°.

Ex. 1357. Show how to inscribe a circle in a given sector.

Ex. 1358. The radius of a circle is 20 ft.; the area of a sector of that circle is 25π sq. ft. What is its arc in degrees?

Ex. 1359. The area of a sector is 18π, the angle of the sector is 80°. Find the radius of the sector.

Ex. 1360. The altitude of an equilateral triangle is 6 in. Find the area of the circumscribed and inscribed circles.

Ex. 1361. The areas of two circles are as 9 : 4 ; the radius of the larger is 6 in. Find the circumference of the smaller circle.

Ex. 1362. Find the area of a circle inscribed in a square whose area is $12\frac{1}{2}$ sq. in.

Ex. 1363. If the apothem of a regular hexagon is 4, what is the area of its circumscribed circle ?

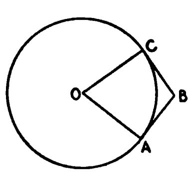
Ex. 1364. Construct a circumference equal to the difference between two given circumferences.

Ex. 1365. It requires $41\frac{1}{7}$ rd. of fencing to inclose a semicircular field. Find the area of the field. (Assume $\pi = 3\frac{1}{7}$.)

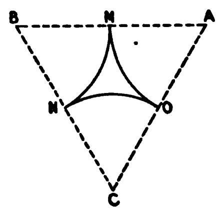
Ex. 1366. Prove that the star-shaped polygon formed by producing the sides of a regular hexagon is equivalent to twice the hexagon.

Ex. 1367. If AB and BC are tangents, $\angle COA = 60°$, and $OA = 10$ in. Find (*a*) area $OABC$, (*b*) area bounded by arc AC and the two tangents.

Ex. 1368. Find the area bounded by the arcs MN, NO, and OM, if each arc $= 60°$ and their common radius equals 10.



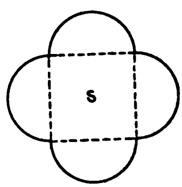
Ex. 1367



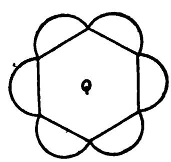
Ex. 1368

Ex. 1369. On the sides of a square whose side is 4 in., semicircles are constructed. Find the total area.

Ex. 1370. On the sides of a regular hexagon whose side equals 2 in. semicircles are drawn. Find the total area.



Ex. 1369



Ex. 1370

Ex. 1371. Find the radius of a circle whose circumference numerically equals its area.

Ex. 1372. Find the central angle of a sector whose area is equal to the square of the radius.

[For practical applications see pp. 303 and 304.]

MISCELLANEOUS EXERCISES

Ex. 1373. Find the radius, the apothem, and the area of the following *regular* polygons:

(*a*) A square whose side equals 4.
(*b*) A square whose side equals s.
(*c*) A hexagon whose side equals 2.
(*d*) A hexagon whose side equals s.
(*e*) A triangle whose side equals 6.
(*f*) A triangle whose side equals s.

Ex. 1374. Find the radius of a decagon whose side equals 8 in.

Ex. 1375. Find the radius of an octagon whose side equals 4 in.

Ex. 1376. The side of the regular circumscribed triangle is equal to twice the side of the regular inscribed triangle.

Ex. 1377. The square of a side of an inscribed equilateral triangle is equal to three times the square of a side of the inscribed regular hexagon.

Ex. 1378. If in a circle two chords intersect at right angles, and circles are constructed on the segments of the chords as diameters, the area of the given circle is equivalent to the sum of the areas of the four circles. (Ex. 1235.)

Ex. 1379. If in a circle a regular decagon and a regular pentagon be inscribed, the side of the decagon increased by the radius is equal to twice the apothem of the pentagon.

Ex. 1380. If the radius of a regular pentagon equals unity, prove that its side equals $\frac{1}{2}\sqrt{10-2\sqrt{5}}$.

HINT. In formula of (439) substitute $n = 5$, then $s_n = s_5 = x$, and $s_{2n} = s_{10}$, whose value may be taken from Ex. 1303.

Ex. 1381. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

* **Ex. 1382.** The square of the side of a regular decagon increased by the square of the radius is equal to the square of the side of a regular pentagon, having the same radius.

Ex. 1383. If two equal lines are divided externally so that the product of the segments of one is equal to the product of the segments of the other, the segments are equal respectively.

* **Ex. 1384.** Two triangles are equal if the base, the opposite angle, and its angle bisector of one are respectively equal to the base, the opposite angle, and its angle bisector of the other.

HINT. Circumscribe circles, produce bisectors until they meet the circumference, and join the point of intersection to one end of the base. (Exs. 1060 and 1383.)

Ex. 1385. If two angle bisectors of a triangle are equal, the triangle is isosceles. (Ex. 1303.)

* **Ex. 1386.** The square of the side of a regular pentagon increased by the square of one of its diagonals is equal to five times the square of the radius. (Ex. 1381.)

* **Ex. 1387** A regular pentagon is equivalent to a rectangle having one side equal to five times the radius, and the other to $\frac{1}{2}$ of a diagonal of the pentagon.

* **Ex. 1388.** The product of the diagonals of an inscribed quadrilateral is equal to the sum of the products of the opposite sides. (Ptolemy's Theorem.)

Ex. 1389. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

Ex. 1390. If circles be circumscribed about the four triangles into which a quadrilateral is divided by its diagonals, their centers form the vertices of a parallelogram.

* **Ex. 1391.** Each angle formed by joining the feet of the three altitudes of a triangle is bisected by the corresponding altitude. (Ex. 581.)

* **Ex. 1392.** If from any point in the circumference of a circle perpendiculars be dropped upon the sides of an inscribed triangle (produced, if necessary), the feet of the perpendiculars are in a straight line. (Ex. 581.)

Ex. 1393. To construct a triangle having given the feet of its three altitudes.

* **Ex. 1394.** If p_n and P_n are respectively the perimeters of an inscribed and circumscribed regular polygon of n sides, and p_{2n}, and P_{2n} the perimeters of regular polygons of $2n$ sides respectively inscribed and circumscribed about the same circle, prove that P_{2n} is the harmonical mean between p_n and P_n; *i.e.*

$$P_{2n} = \frac{2\,p_n P_n}{p_n + P_n}.$$

* **Ex. 1395.** Using the notations of Ex. 1394, prove that p_{2n} is the mean proportional between p_n and P_{2n}; *i.e.* $p_{2n} = \sqrt{p_n P_{2n}}$.

APPENDIX TO PLANE GEOMETRY

ALGEBRAIC SOLUTIONS OF GEOMETRICAL PROPOSITIONS. MAXIMA AND MINIMA PRACTICAL PROBLEMS

CONSTRUCTION OF ALGEBRAIC EXPRESSIONS

442. Note. In the following propositions, a, b, c, d, etc., denote given lines, while x, y, z, etc., denote required lines.

443. (1) Construct $x = a + b$.

(2) Construct $x = a - b$.

(3) If m denotes a given number.

Construct $x = m \cdot a$.

(4) Construct $x = \frac{a}{m}$.

(5) Construct $x = \frac{ab}{c}$.

Hint. x is the fourth proportional to c, a, and b.

(6) Construct $x = \frac{a^2}{b}$.

(7) Construct $x = \sqrt{ab}$.

Hint. x is the mean proportional between a and b (325).

(8) Construct $x = \sqrt{a^2 + b^2}$.

(9) Construct $x = \sqrt{a^2 - b^2}$.

(10) Construct $x = a\sqrt{2}$.

(11) If m denotes a known number.

Construct $x = a\sqrt{m}$.

Hint. $x = \sqrt{a(am)}$.

Ex. 1396. Construct $x = \frac{a}{2}\sqrt{3}$ by means of an equilateral triangle.

444. Complex algebraic expressions are constructed by means of the eleven constructions of (443). It is quite often necessary to transform the algebraic expressions, in order to make them special cases of (443). Different algebraic transformations lead to different solutions.

Construct:

Ex. 1397.
$$x = \sqrt{3\,ab},$$
$$x = \sqrt{(3\,a)b},$$
i.e. x is the mean proportional between $3\,a$ and b,

or
$$x = \sqrt{ab}\sqrt{3},$$
i.e. find $\sqrt{ab}$ by means of (443, 7), and $\sqrt{ab}\sqrt{3}$ by means of (Ex. 1396).

Ex. 1398.
$$x = \frac{a^2 - b^2}{4\,c} = \frac{(a+b)(a-b)}{4\,c},$$
i.e. find the fourth proportional to $4\,c$, $a+b$, and $a-b$.

Ex. 1399.
$$x = \sqrt{a^2 - ab}.$$
HINT.
$$x = \sqrt{a(a-b)}.$$

Ex. 1400.
$$x = \sqrt{a^2 - bc} = \sqrt{a^2 - (\sqrt{bc})^2}.$$
i.e. find the mean proportional between b and c, and construct a right triangle, having one arm equal to the mean proportional, and the hypotenuse equal to a.

445. REMARK. The expressions to be constructed in Geometry are always homogeneous, and either of the first or second degree.

Some expressions may be made homogeneous by the introduction of unity. Thus $x = ab$, may be written $x = \frac{ab}{1}$, *i.e.* x is the fourth proportional to 1, a, and b.

446. **The impossibility of the construction** is indicated if:

(1) The expression is imaginary.

(2) The value of x is not within the limits indicated by the problem.

(3) Sometimes, if the result is negative. If the required line has a certain direction, a negative result would indicate a *line* of opposite direction.

Ex. 1401. Construct the following expressions:

(a) $\sqrt{2ab}$, (b) $a\sqrt{5}$, (c) $a+3b$, (d) $\sqrt{4a^2-b^2}$.

(e) $\frac{ab+cd}{e}$, $\left(i.e.\ \frac{ab}{e}+\frac{cd}{e}\right)$.

(f) $\frac{4a^2-b^2}{c}$, $\left(i.e.\ \frac{(2a+b)(2a-b)}{c}\right)$.

(g) $\frac{abc}{de}$, $\left[i.e.\ \frac{a\cdot\frac{bc}{d}}{e}\right]$.

(h) $\sqrt{a^2-ac}$, (i) $\frac{a\sqrt{bc}}{d}$, (k) $\sqrt{a^2+2ab+b^2-c^2}$.

(l) $\sqrt{\frac{a^2}{4}+b^2}$.

(m) $\sqrt{9a^2-bc}$, (i.e. $\sqrt{(3a)^2-(\sqrt{bc})^2}$).

(n) $a\sqrt[4]{3}$, (i.e. $\sqrt{a\cdot\sqrt{a(3a)}}$).

(o) $\sqrt[4]{abcd}$, (i.e. $\sqrt{\sqrt{ab}\sqrt{cd}}$).

SOLUTION OF PROBLEMS BY MEANS OF ALGEBRAIC ANALYSIS *

447. If a problem requires the construction of lines, it is often possible to state the condition in the form of an equation. The solution of the equation gives the unknown line in an algebraic form, which may be constructed according to (444).

Ex. 1402. To divide a line externally so that the small external segment is the mean proportional between the other segment and the given line.

Analysis. Let x be one segment, then $a+x$ is the other, and

$$a+x:x=x:a.$$

$$x^2=a^2+ax.$$

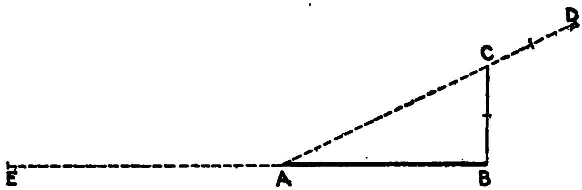
The solution of the equation gives

$$x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+a^2}.$$

* This chapter presupposes the knowledge of quadratic equations.

As the minus sign would give a negative answer, *i.e.* an internal segment, we have to construct

$$x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + a^2}.$$



Construction.

Draw $AB = a$. At B, draw BC perpendicular to AB and equal to $\frac{a}{2}$.

Draw AC, and produce it to D, so that $CD = CB$.

Then $x = AD$.

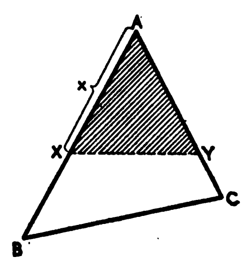
$\therefore$ produce BA to E, so that $AE = AD$.

E is the required point.

448. The analysis used in this problem is called **algebraic analysis** in distinction from the purely **geometric analysis** given in Book I.

The algebraic analysis contains a proof for the correctness of the construction, although quite often a purely geometric proof may be found.

Ex. 1403. From a triangle ABC, to cut off an isosceles triangle AXY, equivalent to one half of ABC.



Analysis. Let

$$AX = AY = x,$$

$$AB = c,$$

and

$$AC = b.$$

$$\triangle AXY : \triangle ABC = x^2 : bc = 1 : 2. \quad \text{(Why?)}$$

$$\therefore 2x^2 = bc.$$

$$x = \sqrt{\frac{bc}{2}}.$$

Construction. Construct x, a mean proportional between $\frac{c}{2}$ and b, and on AB and AC respectively, lay off AX and AY equal to x.

Then $\triangle AXY$ is the required one.

Ex. 1404. To construct a right triangle, having given an arm a, and the projection p, of the other arm upon the hypotenuse.

Analysis. Let x be the projection of a upon the hypotenuse.

$$x(x+p)=a^2, \qquad \text{(Why?)}$$

or

$$x^2+xp=a^2.$$

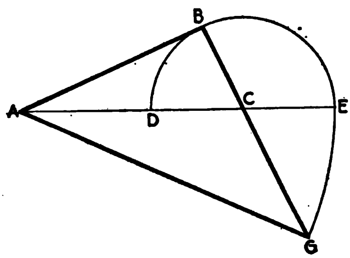
$$\therefore x=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2+a^2}.$$

The minus sign would give a negative answer.

$$\therefore \text{construct } x=-\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+a^2}.$$

Construction. Draw $AB=a$, and BC perpendicular to AB and equal to $\frac{1}{2}p$. Draw AC.

$$AC=\sqrt{\left(\frac{p}{2}\right)^2+a^2}.$$



From C as a center, with a radius equal to BC, draw an arc intersecting AC in D and E.

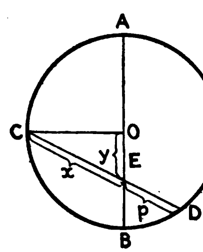
Then $AD=x$,

and $AE=x+p=$ hypotenuse.

$\therefore$ from A as a center, with a radius equal to AE, draw an arc meeting BC produced in G.

AGB is the required triangle.

Ex. 1405. In a circle O a diameter $AB=2r$ is drawn. From the mid-point C of the semicircle AB to draw a chord CD, meeting AB in E, so that DE shall be equal to a given line p.



Analysis. Let

$$CE=x,\ OE=y.$$

$$x^2=y^2+r^2, \qquad \text{(Why?)}$$

and

$$x\cdot p=(r+y)(r-y). \qquad \text{(Why?)}$$

Eliminate y and solve. Then

$$x=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2+2r^2}.$$

Only the positive sign gives a line.

Construct $$x = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + 2r},$$

or transformed, $$x = \sqrt{\left(\frac{p}{2}\right)^2 + (r\sqrt{2})^2} - \frac{p}{2}.$$

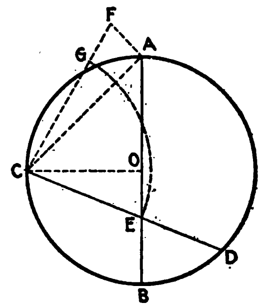
Construction. Draw $CA (= r\sqrt{2})$.

At A, draw AF perpendicular to CA and equal to $\frac{p}{2}$.

Draw CF. On FC, lay off FG equal to FA.

Then $x = CG$.

From C as a center, with a radius equal to CG, draw an arc meeting AB in E. Through E, draw chord CD, which is the required chord.



Ex. 1406. Given lines AB and p. In the line AB, to find a point C so that $\overline{AC}^2 - \overline{CB}^2 = p^2$.

Ex. 1407. In the line AB, to find a point C so that $\overline{AC}^2 = 2\,\overline{CB}^2$.

Ex. 1408. To divide the larger side of a given rectangle into two parts so that the difference of their squares equals the area of the rectangle.

Ex. 1409. Given a line m. To construct a right $\triangle\, abc$ (c being the hypotenuse) so that $c - b = b - a = m$.

Ex. 1410. In the median AD, drawn to the base of isosceles $\triangle\, ABC$, to find a point X such that XD and the perpendiculars dropped from X upon AB and AC divide the figure into three equivalent parts.

Ex. 1411. From a point P without a circumference, to draw a secant which is bisected by the circumference.

Ex. 1412. In a given square, to inscribe another square, having a given side.

Ex. 1413. To inscribe a square in a semicircle.

Ex. 1414. In the triangle ABC, to inscribe a parallelogram equivalent to $\frac{1}{3}$ ABC and having an angle common with $\triangle\, ABC$.

Ex. 1415. Upon a given line as hypotenuse to construct a right triangle one of whose arms is a mean proportional between the other arm and the hypotenuse.

Ex. 1416. To transform a given square into a rectangle having a perimeter equal to twice the perimeter of the given square.

Ex. 1417. Given two concentric circles. To draw a chord in the larger circle so that it equals twice the chord formed in the smaller circle.

***Ex. 1418.** To construct a triangle, having given the base, the vertex angle, and the bisector of that angle.

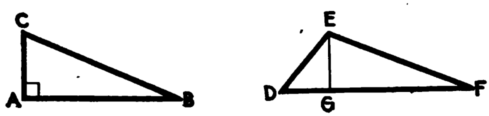
MAXIMA AND MINIMA OF PLANE FIGURES

449. Def. The greatest of all magnitudes that satisfy given conditions is called the **maximum**; the least is called the **minimum.**

450. Def. **Isoperimetric figures** are those which have **equal perimeters.**

Proposition I. Theorem

451. *Of all triangles having given two sides, that in which those two sides are perpendicular to each other is a maximum.*



Given in $\triangle$ ABC and DFE, $AB = DF$, $AC = DE$, $\angle A =$ rt. $\angle$, and $\angle D$ oblique.

To prove area $ABC >$ area DEF.

Proof. From E draw $EG \perp DF$.

Then $EG < DE$, (Why ?)

or $EG < AC$.

$\therefore$ $\triangle$ ABC and DEF have equal bases and unequal altitudes.

$\therefore \triangle ABC > \triangle DEF$. (Why ?)

$\therefore \triangle ABC$ is a maximum. Q. E. D.

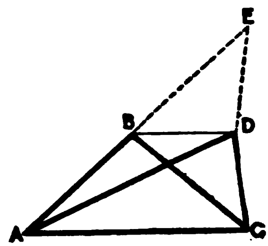
Ex. 1419. Of all triangles having given the base and the median to the base, which is the maximum ?

Ex. 1420. Of all parallelograms having given the diagonals, which has the maximum area ?

Ex. 1421. To divide a given line into two parts such that the rectangle contained by the segments is a maximum.

PROPOSITION II. THEOREM

452. *Of all triangles having the same base and equal areas, the isosceles triangle has the minimum perimeter.*



Given $\triangle ADC = \triangle ABC$, and $AB = BC$.

To prove $AB + BC + AC < AD + DC + AC$.

Proof. Produce AB by its own length to E. Draw BD and DE.

$BD \parallel AC$,

(*for otherwise* $\triangle ABC$ *would not be* $= \triangle ADC$).

$\angle EBD = \angle BAC$. (Why?)

$\angle BAC = \angle BCA$. (Why?)

$\angle BCA = \angle CBD$. (Why?)

$\therefore \angle EBD = \angle CBD$. (Ax. 1.)

But $BC = BE$, (Why?)

and BD is common.

$\therefore \triangle BDC \cong \triangle BDE$.

$\therefore DC = DE$.

But $AD + DE > AB + BE$. (Why?)

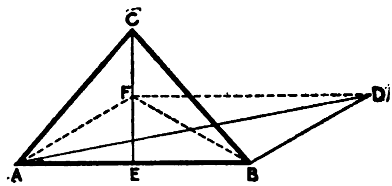
$\therefore AD + DC > AB + BC$,

or adding AC,

$AC + AD + DC > AB + BC + AC$. Q. E. D.

PROPOSITION III. THEOREM

453. *The maximum of isoperimetric triangles on the same base is the one whose other two sides are equal.*



Given $\triangle ABC$ and ABD having equal perimeters, the common base AB, and $AC = CB$.

To prove area $ACB >$ area ADB.

Proof. Draw median CE and $DF \parallel AB$, meeting CE in F. Draw FA and FB.

Then CE is the perpendicular bisector of AB. (Why?)

$$\therefore AF = FB$$

But $\triangle AFB = \triangle ADB$. (360)

$\therefore$ perimeter $AFB <$ perimeter ADB. (452)

$$\therefore AF + FB < AD + DB.$$

But $AB + DB + DA = AB + BC + CA$. (Hyp.)

$\therefore DB + DA = BC + CA$. (Ax. 3.)

$\therefore AF + FB < BC + CA$. (Sub.)

But $AF = FB$ and $BC = CA$.

Hence $AF < AC$.

$\therefore FE < CE$. (327)

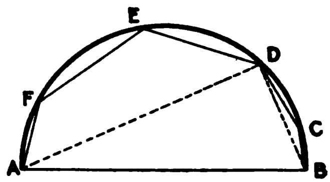
$\therefore$ area $AFB <$ area ACB, (358)

or area $ADB <$ area ACB. Q. E. D.

454. COR. Of all isoperimetric triangles, the equilateral has the maximum area.

PROPOSITION IV. THEOREM

455. *Of all polygons having all sides given but one, the maximum can be inscribed in a semicircle having the undetermined side as diameter.*



Given polygon $ABCDEF$ the maximum of all polygons having given the sides AF, FE, ED, DC, and CB.

To prove $ABCDEF$ can be inscribed in a semicircle whose diameter is AB.

Proof. Join any vertex, as D, with A and B.

Then $\triangle ADB$ must be the maximum of all triangles that can be formed with sides AD and DB; *i.e.* $\angle ADB$ must be a rt. $\angle$. For otherwise by making $\angle ADB$ a right one without changing the sides AD and DB, we could increase $\triangle ADB$ without altering the remaining parts $AFED$ and DCB of the polygon. Or polygon $ABCDEF$ would be increased, which is contrary to the hypothesis, since $ABCDEF$ is a maximum.

$\therefore \angle ADB$ is a right angle,

and D is on a semicircumference that can be constructed on AB.

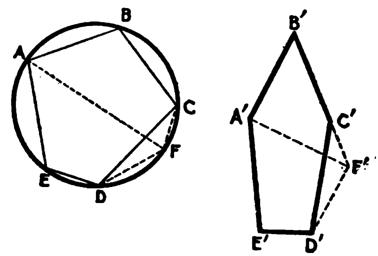
For the same reason every vertex of the polygon must lie on the semicircumference. Q. E. D.

Ex. 1422. In quadrilateral $ABCD$, $AB = BC = CD =$ a given line. Determine the angles A, B, C, and D so that $ABCD$ becomes a maximum.

Ex. 1423. To inscribe an angle in a given semicircle so that the sum of its arms is a maximum.

PROPOSITION V. THEOREM

456. *Of all polygons constructed with the same given sides, that which can be inscribed in a circle is the maximum.*



Given polygon $ABCDE$ inscribed in a circle, mutually equilateral with polygon $A'B'C'D'E'$, which cannnot be inscribed in a circle.

To prove area $ABCDE >$ area $A'B'C'D'E'$.

Proof. From A draw diameter AF, and join F to the two nearest vertices C and D.

On $C'D'$, construct $\triangle C'D'F'$ equal to $\triangle CDF$, and join $A'F'$.

Area $ABCF >$ area $A'B'C'F'$. (455)

Area $FDEA >$ area $F'D'E'A'$. (455)

$\therefore$ area $ABCFDE >$ area $A'B'C'F'D'E'$.

But $\triangle CFD \cong \triangle C'F'D'$.

Then area $ABCDE >$ area $A'B'C'D'E'$. (Ax. 5.)

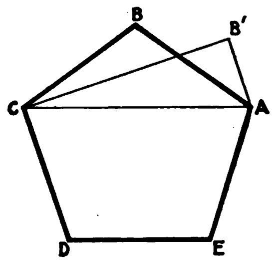
Q. E. D.

Ex. 1424. In quadrilateral $ABCD$, $AB = 2$ inches, $BC = CD = DA = 1$ inch. What must be the angles A, B, C, and D to make the area $ABCD$ a maximum?

Ex. 1425. Determine the angles of pentagon $ABCDE$, if its area is a maximum and $AB = BC = CD = 2$ inches, and $DE = EF = 2\sqrt{2}$ inches.

PROPOSITION VI. THEOREM

457. *Of all isoperimetric polygons of the same number of sides, the equilateral is the maximum.*



Given $ABCDE$ the maximum of all isoperimetric polygons of the same number of sides.

To prove $AB = BC = CD = DE = EA$.

Proof. Suppose polygon $AB'CDE$, containing the two unequal sides CB' and AB' were the maximum.

Then construct on the diagonal AC the isosceles triangle $AB'C$, isoperimetric with ABC.

Area $AB'C$ is smaller than area ABC. (453)

$\therefore$ polygon $AB'CDE$ is smaller than the isoperimetric polygon $ABCDE$.

Hence polygon $AB'CDE$ cannot be the maximum.

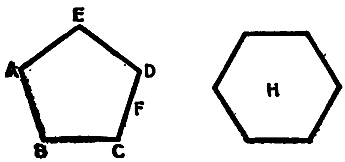
Or in the maximum polygon $AB = BC$. Therefore all sides of the maximum polygon are equal. Q. E. D.

458. COR. Of all isoperimetric polygons of the same number of sides, **the maximum is regular.**

Ex. 1426. In a given semicircle to inscribe a trapezoid whose area is a maximum.

Proposition VII. Theorem

459. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Given regular pentagon $ABCDE$ isoperimetric with regular hexagon H.

To prove area $H >$ area $ABCDE$.

Proof. Let F be any point in CD.

$ABCFDE$ may be considered a hexagon, having one of its angles equal to a straight angle.

$ABCFDE$ is not an equilateral polygon. H is an equilateral polygon.

$\therefore$ area $H >$ area $ABCDE$. (457)

Q. E. D.

460. Cor. The area of a circle is greater than the area of any polygon whose perimeter equals the circumference of the circle.

Ex. 1427. In a given segment to inscribe an angle so that the sum of its arms is a maximum.

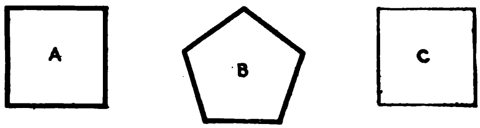
Ex. 1428. In a given circle to inscribe a triangle, having a maximum perimeter.

Ex. 1429. The circumscribed regular polygon has a smaller area than any other polygon circumscribed about the same circle.

Ex. 1430. Of all equivalent parallelograms on the same base, which has the minimum perimeter ?

PROPOSITION VIII. THEOREM

461. *Of two equivalent regular polygons, that which has the greater number of sides has the smaller perimeter.*



Given square A = regular pentagon B.

To prove perimeter of $A >$ perimeter of B.

Proof. Construct square C isoperimetric with B.

Area $C <$ area B, (449)

or area $C <$ area A.

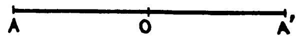
$\therefore$ perimeter of $C <$ perimeter of A.

$\therefore$ perimeter of $B <$ perimeter of A. Q.E.D.

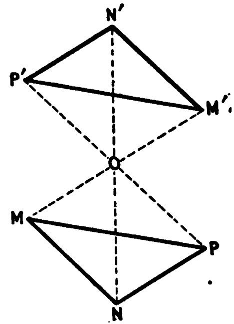
462. COR. The circumference of a circle is less than the perimeter of any equivalent polygon.

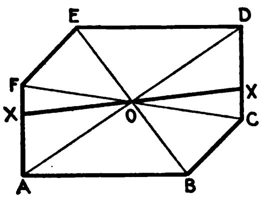
SYMMETRY

463. Two points A and A' are **symmetric with respect to a point** O, if O is the mid-point of AA'.



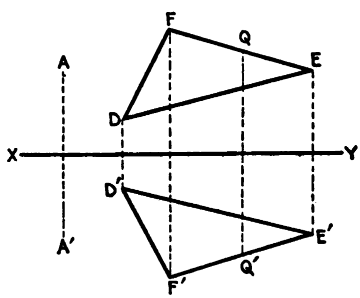
464. A figure is **symmetric with respect to a point** O, if any point in the figure is symmetric to another point in the figure with respect to O.





Thus, $ABCDEF$ is a figure symmetric to O. Obviously any line drawn through O and terminated by the figure, as XOX', is bisected. If the figure consists of two parts, as $\triangle MNP$ and $\triangle M'N'P'$, the two parts (*i.e.* the two triangles) are symmetric with regard to O.

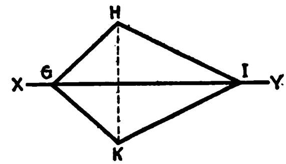
465. Two points A and A' are **symmetric with regard to an axis** XY, if XY is the perpendicular-bisector of AA'.



466. Two figures are **symmetric with regard to an axis** XY, if to every point in the figure Q, there exists a corresponding point in the other, Q', so that Q and Q' are symmetric with respect to XY.

Thus, $\triangle DEF$ and $\triangle D'E'F'$ are symmetric with respect to XY.

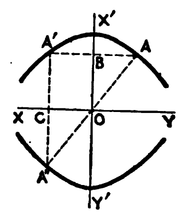
The two figures may also form parts of one figure, as $GHIK$. Then the entire figure ($GHIK$) is said to be symmetric with respect to XY.



467. If two figures I and II are symmetric with regard to an axis XY and one figure I is turned about XY until it lies on the same side of XY as the other figure II, then I and II will coincide.

Conversely, if such a rotation produces a coincidence of the figures, they are symmetric with respect to XY.

468. THEOREM. If a figure is symmetric with respect to two axes perpendicular to each other, it is symmetric with respect to their point of intersection.



Let A be a point in the figure. There must exist another point A' so that $AA' \perp X'Y'$ and $AB = A'B$. Similarly there must be another point A'' so that $A'A'' \perp XY$ and $A'C = A''C$. Obviously $AA' \parallel XY$ and $A'A'' \parallel XY$.

Draw AA''.

Then $X'Y'$ bisects AA''. (152)

Similarly XY bisects AA''. (152)

$\therefore$ XY and $X'Y'$ meet AA'' in its mid-point.

Or line AA'' is bisected by O, the point of intersection of XY and $X'Y''$. Or to any point (A) there exists another A'' so that A and A'' are symmetric with regard to O.

Ex. 1431. A parallelogram is symmetric with respect to the point of intersection of its diagonals.

Ex. 1432. An isosceles triangle is symmetric with respect to the median drawn to the base.

Ex. 1433. Quadrilateral $ABCD$ is symmetric with respect to the diagonal AC, if $AB = AD$, and $BC = CD$.

Ex. 1434. A regular polygon is symmetric with respect to its center.

Ex. 1435. Two figures symmetric with respect to an axis are congruent.

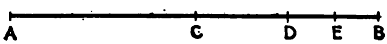
Ex. 1436. Two figures symmetric with respect to a point are congruent.

THE INCOMMENSURABLE CASE BASED UPON LIMITS *

469. A **constant** is a quantity that maintains the same value throughout the same discussion. A **variable** is a quantity whose value changes during the same discussion.

Two quantities are **commensurable** if they have a common measure. Two quantities are **incommensurable** if they have no common measure.

470. If a variable x approaches a constant a so that the difference between a and x becomes less than any conceivable number, then a is called the **limit** of x.



For example, suppose a point P to move from A to B in such a way as to move in the first second over half of AB to C, in the second second, over half of the remainder, CB, to D, in the third second over half of the new remainder, DB, to E, and so on indefinitely.

It is evident that the distance from A to the moving point P is a variable whose value can be made to differ from AB by less than any assigned quantity, although it never can be made equal to AB.

AB is therefore the **limit of the variable.**

Ex. 1437. What is the limit of .999 ⋯?

Ex. 1438. What is the limit of $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$ to infinity?

471. THEOREM. **If two variables, x and y, are always equal and x approaches a as a limit, then y approaches a as a limit.**

Proof. $a - x = a - y$, and $a - x$ can be made less than any assignable number.

Hence, $a - y$ can be made smaller than any assignable number.

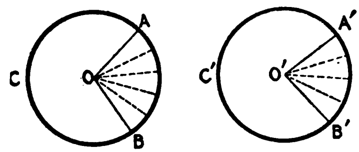
$\therefore$ y approaches a as a limit.

472. COR. **If two variables are always equal and each approaches a limit, the limits are equal.**

* For greater detail of limits see §§ 418 to 422.

Proposition I. Theorem

473. *In the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs.*



Given in the equal circles ABC and $A'B'C'$, two central angles AOB and $A'O'B'$, intercepting the arcs AB and $A\ B'$ respectively.

To prove
$$\frac{\angle AOB}{\angle A'O'B'} = \frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}.$$

Proof. Case I. When the arcs are commensurable.

Let m be a common measure contained in AB five times and in $A'B'$ four times.

Then,
$$\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \frac{5}{4}.$$

Connect the points of division with the center. Then $\angle AOB$ will have been divided into 5 parts and $\angle A'O'B'$ into four, all being equal. (183)

Whence
$$\frac{\angle AOB}{\angle A'O'B'} = \frac{5}{4}.$$
Q. E. D.

Hence if
$$\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \frac{5}{4}, \text{ then}$$

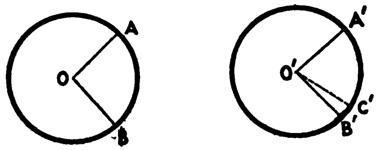
$$\frac{\angle AOB}{\angle A'O'B'} = \frac{AB}{A'B'}. \qquad \text{(Ax. 1.)}$$

But obviously, this conclusion can be demonstrated in like manner if $\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}$ equals any other fraction. Hence,

$$\frac{\angle AOB}{\angle A'O'B'} = \frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}.$$
Q. E. D.

CASE II. When the arcs are incommensurable.

Divide $\widehat{AB}$ into any number of equal parts, and apply one of those parts to $\widehat{A'B'}$ as many times as possible.



Since $\widehat{AB}$ and $\widehat{A'B'}$ are incommensurable, there must be a remainder $C'B'$ less than one of the equal parts.

Draw $O'C'$.

Since the arcs AB and $A'C'$ are commensurable,

$$\frac{\widehat{A'C'}}{\widehat{AB}} = \frac{\angle A'O'C'}{\angle AOB}.$$

By increasing the number of parts into which AB is divided, we can diminish the length of each part, and, therefore, the length of $C'B'$ indefinitely.

Hence $\widehat{A'C'}$ approaches $\widehat{A'B'}$ as a limit, and $\angle A'O'C'$ approaches $\angle A'O'B'$ as a limit.

Therefore $\frac{\widehat{A'C'}}{\widehat{AB}}$ approaches $\frac{\widehat{A'B'}}{\widehat{AB}}$ as a limit, and $\frac{\angle A'O'C'}{\angle AOB}$ approaches $\frac{\angle A'O'B'}{\angle AOB}$ as a limit.

The variables $\frac{\widehat{A'C'}}{\widehat{AB}}$ and $\frac{\angle A'O'C'}{\angle AOB}$ being always equal, must have equal limits. (472)

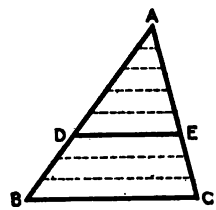
Whence $$\frac{\widehat{A'B'}}{\widehat{AB}} = \frac{\angle A'O'B'}{\angle AOB}.$$

Q. E. D.

474. COR. A central angle is measured by its intercepted arc.

PROPOSITION II. THEOREM

475. *A line parallel to one side of a triangle divides the other two sides proportionally.*



Given in $\triangle ABC$, DE parallel to BC.

To prove $AD : DB = AE : EC$.

Proof. CASE I. AD and DB are commensurable.

Let m be a common measure contained in AD a times and in DB b times.

Then $$\frac{AD}{DB} = \frac{a}{b}.$$

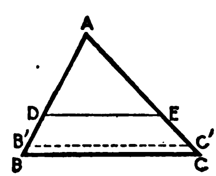
Through the points of division of AB draw parallels to BC. These lines divide AE into a parts and EC into b parts, all being equal. (152)

Whence $$\frac{AE}{EC} = \frac{a}{b}.$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \qquad \text{(Ax. 1.)}$$

Q.E.D.

CASE II. AD and DB are incommensurable.



Divide AD into any number of equal parts, and apply one of those parts to DB as many times as possible.

As the lines AD and DB are incommensurable, there must be a remainder, $B'B$, less than one of the equal parts.

Draw $B'C'$ parallel to BC.

The lines AD and DB' are commensurable.

$$\therefore \frac{AD}{DB'} = \frac{AE}{EC'}.$$

By increasing the number of parts into which AD is divided, we can diminish the length of these parts, and therefore the length of $B'B$ indefinitely.

Hence DB' approaches DB as a limit, and EC' approaches EC as a limit.

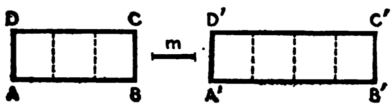
The variables $\frac{AD}{DB'}$ and $\frac{AE}{EC'}$, being always equals, must have equal limits.

Whence $$\frac{AD}{DB}=\frac{AE}{EC}. \qquad (472)$$

Q. E. D.

Proposition III. Theorem

476. *Rectangles having equal altitudes are to each other as their bases.*



Given in rectangles $ABCD$ and $A'B'C'D'$, altitude $AD =$ altitude $A'D'$.

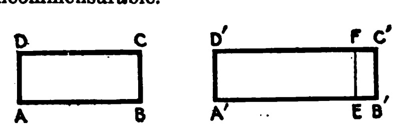
To prove $$\frac{ABCD}{A'B'C'D'}=\frac{AB}{A'B'}.$$

Proof. Case I. AB and $A'B'$ are commensurable.

Let m be a common measure contained in AB a times, and in $A'B'$ b times.

[To be completed by the student. Compare (475).]

Case II. AB and $A'B'$ are incommensurable.



Divide AB into any number of equal parts, and lay off one of these parts on $A'B'$ as many times as possible.

As AB and $A'B'$ are incommensurable, there must be a remainder, EB', less than one of the equal parts.

Draw $EF \perp A'B'$.

As AB and $A'E$ are commensurable,

$$\frac{ABCD}{A'EFD'}=\frac{AB}{A'E}.$$

By increasing the number of parts into which AB is divided, we can diminish the length of these parts, and, therefore, the length of EB' indefinitely.

Hence, $A'E$ approaches $A'B'$ as a limit, and $A'EFD'$ approaches $A'B'C'D'$ as a limit.

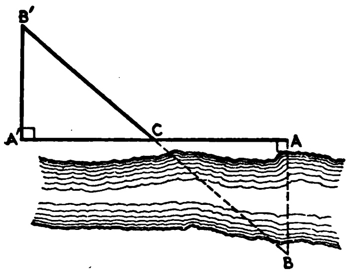
The variables $\frac{ABCD}{A'EFD'}$ and $\frac{AB}{A'E}$ being always equal, must have equal limits.

Whence $$\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}.$$

Q. E. D.

PRACTICAL APPLICATIONS OF PLANE GEOMETRY

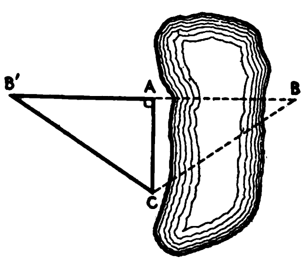
BOOK I *



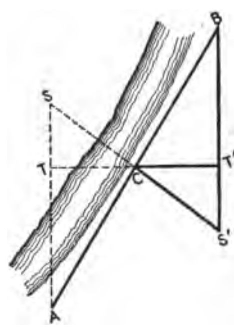
1. In order to determine the distance across a river AB, we measure AA' at right angles to AB, and place a stake at C, the mid-point of AA'. From A' we walk in a direction perpendicular to AA' until we reach the point B', which is in a straight line with B and C. Which distance must we measure in order to obtain the width of the river? Why?

'2. Show what line we have to measure in order to determine the width of a lake AB, if we make $\angle ACB = \angle ACB'$ and $BB' \perp AC$.

3. In order to measure the distance of a tree T from a steeple S, both on the other side of a river, I locate a point A in the



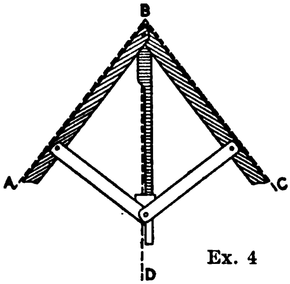
Ex. 2



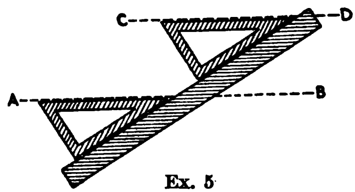
Ex. 3

* It is strongly recommended that students measure some angles in the field by means of a transit, a plane table, a sextant, etc. It is not necessary that such instruments be very elaborate, and even very crude ones, made of two paper protractors, a ruler, and a few pins, are serviceable.

prolongation of ST, take a line AB, and place a stake at its mid-point C. From B I walk in a direction BS' so that $\angle B = \angle A$, and put stakes at the point T' which lies in the prolongation of TC, and at the point S', which lies in the prolongation of SC. Which line must I measure? Prove your statement.



Ex. 4

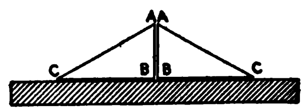


Ex. 5

4. Show how the annexed *Angle Divider* can be used to obtain the bisector BD of any angle ABC.

5. Show how a ruler and a triangle can be used to draw a line CD parallel to a given line AB.

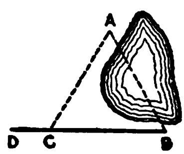
6. A triangle is placed against a ruler in the two positions which are indicated in the diagram, and the position of AB is in both cases the same. How many degrees are in angle B?



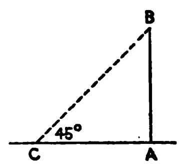
7. To measure the distance AB, we walk from B toward D so that $\angle B = 60°$, until we reach C, a point at which

$$\angle ACB = 60°.$$

What line must we measure to obtain AB? Why?

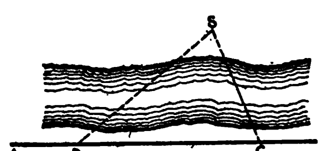


Ex. 7



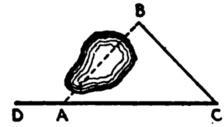
Ex. 8

8. At a point C, 80 ft. from the foot of a pole AB, the angle ACB was found to be 45°. How high is the pole?

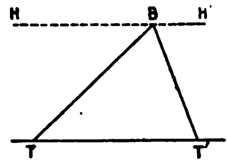


9. Walking along the bank of a river from A to C, a surveyor measures at B the angle ABS, and walking 800 ft. further to C, he found angle BCS to be one half of angle ABS. What is the distance BS?

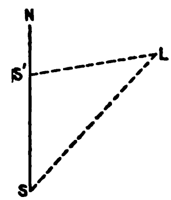
10. Find the length AB of a lake from the following measurements: $\angle DAB = 138°$, $\angle C = 42°$, $AC = 610$ yd., $BC = 400$ yd.



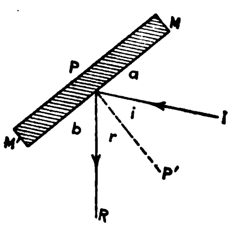
11. A man in a balloon B measures the angles which a horizontal line HH' forms with the lines drawn to two towns T and T' as follows: $\angle HBT = 40°$, $\angle H'OT' = 70°$. If the distance of the towns is two miles, and if the balloon is directly above the line TT', what is the distance of the balloon from T?

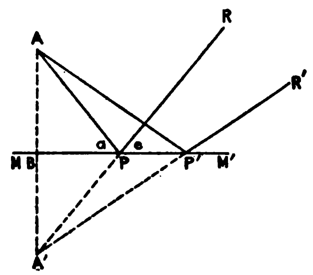


12. A ship sailing north at the rate of 10 mi. per hour occupied at 8 A.M. the position S, and at 10 A.M. the position S'. How far was the ship at 10 A.M. from a lighthouse L, if $\angle S = 43°$ and $\angle NS'L = 86°$?



13. In the diagram given here (and in the diagram of similar exercises) MM' represents a mirror perpendicular to the plane of the paper, and all other lines represent lines *in* the plane of paper.

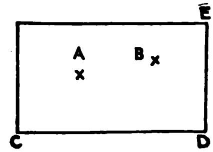




If PP is $\perp MM$, IP is a ray of light and PR is the reflected ray, then $\angle i$ ("angle of incidence") always equals $\angle r$ ("angle of reflection"), hence $\angle a = \angle b$.

If rays emitted from a point A are reflected by a mirror MM', prove that we obtain the reflected rays (*i.e.* make the necessary angles equal) by the following construction: Draw $AB \perp MM'$ and produce AB by its own length to A'. Draw $A'P$. PR, the prolongation of $A'P$, is the reflected ray.

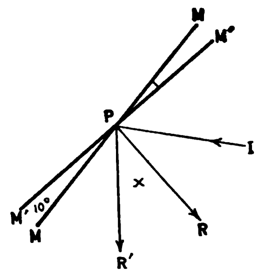
14. A billiard ball is reflected by a cushion according to the same law as a ray of light is reflected by a mirror. *I.e.* the angle of incidence equals the angle of reflection. (Ex. 13.)



The billiard ball A is to hit a point P in the cushion CD, and be reflected therefrom so as to strike B. Determine the location of P by a construction.

15. If the ball a shall first strike cushion CD, then cushion DE, and finally hit B, construct the path of the billiard ball.

16. The ray IP is reflected by MM in the direction PR, and if the mirror is turned 10° in the position MM' ($\angle MPM' = 10°$) the same ray is reflected in the direction PR'. Find $\angle RPR'$.



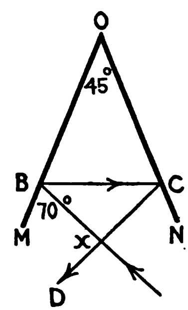
17. Solve the same problem if

$$\angle MPM' = n°.$$

NOTE. The result of this problem forms the principle of the sextant, an instrument used chiefly by sailors, for measuring angles.

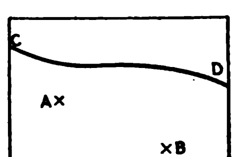
18. Two mirrors MO and NO form an angle O equal to 45°. If a ray AB is reflected in the direction BC, and BC in the direction of CD, find the angle formed by the first and last ray (*i.e.* $\angle x$), if

$$(a)\ \angle ABM = 70°.$$
$$(b)\ \angle ABM = n°.$$



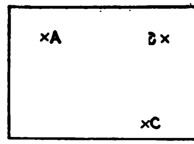
NOTE. The preceding exercise explains the principle of an instrument (called "optical square") which is used to lay off right angles.

19. The annexed diagram represents a portion of a map, A and B two towns, and CD a railroad.

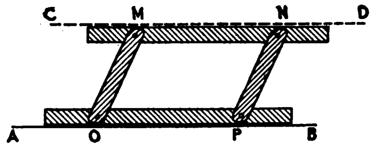


It is proposed to build a station, S, which is just as far from A as from B. Locate S by a construction.

20. A, B, and C represent the locations of three towns (on a map). It is proposed to build a common schoolhouse, S, which is equally far from A, B, and C. By a construction locate S.



21. From a ship which sails N.E. (*i.e.* north-east) at the rate of 6 mi. an hour, a lighthouse is observed. At 9 P.M. the lighthouse appears exactly East, at 11 P.M. it appears exactly South. At what hour was the ship nearest to the lighthouse and what was the distance between them at this hour?

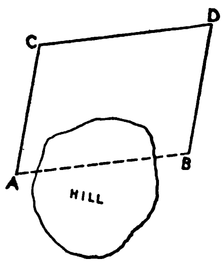


22. Explain how the instrument represented in the figure can be used to draw lines (*e.g.* CD) parallel to a given line AB [M, N, O, and P are pivots, $MO = NP$, and $MN = OP$].

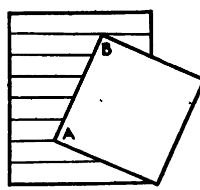
23. To measure the distance from A to B, two points on opposite sides of a hill, lay off two lines, AC and BD, both running due north, and make

$$AC = BD.$$

Which line must be measured to obtain the length of AB? Why?



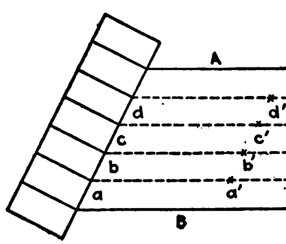
Ex. 23



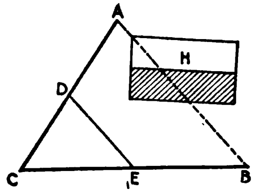
Ex. 24

24. Explain how an ordinary ruled sheet can be used to divide a given line AB in 5 equal parts. Under what conditions is it impossible to use this method?

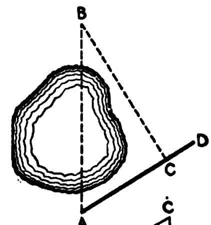
25. To divide a strip AB into 5 equal strips by lines parallel to A and B, apply a ruled sheet as indicated in the figure and obtain the 4 points a, b, c, d. In like manner obtain a', b', c', and d', and join the corresponding points. Prove this construction.



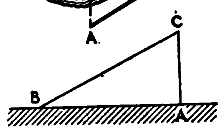
26. Show how to determine the distance between two points A and B, which are separated by a house, by measuring the following lines: $AD = 9$ yd., $AC = 18$ yd., $EC = 11$ yd., $CB = 22$ yd., and $DE = 10$ yd.



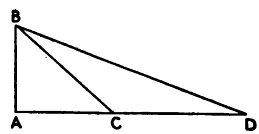
27. To measure the distance AB, a man walks in the direction AD, so that $\angle BAD = 60°$, and stops at a point C, where $\angle BCD = 90°$. If $AC = 200$ yd., what is the length of AB?



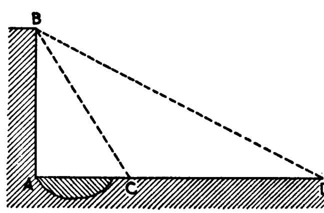
28. The top of a flagstaff, 75 ft. high, is partly broken and touches the ground at B. Find the height of AC, if $\angle B = 30°$.



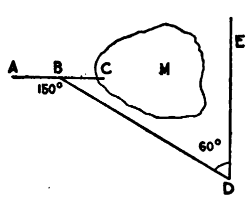
29. To measure the height, AB, of a tower, a man walks on a level straight road DCA that leads directly to the foot of the tower: At D, $\angle BDA = 15°$, and at C, $\angle BCA = 30°$. Find AB, if $DC = 300$ ft.



30. AB is a perpendicular cliff rising at one side of a river. An observer at a point C, 20 ft. from the opposite bank, measures $\angle ACB = 60°$, and walking 200 ft. further away from the river to D, he measures $\angle BDA = 30°$. Find the breadth of the river.

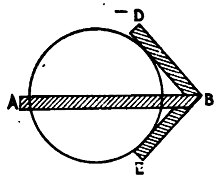


31. A straight railway AB meets a mountain at C, and a tunnel is being driven at C in the direction AC. It is proposed to commence this work at the other side of the mountain also. Angle ABD is made equal to 150°, $BD = 3$ km., and $\angle D = 60°$. How far from D in DE must the tunnel be driven, and what is its direction?

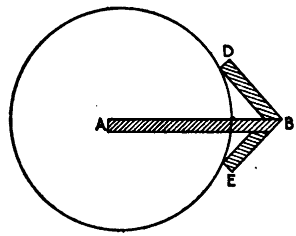


BOOK II

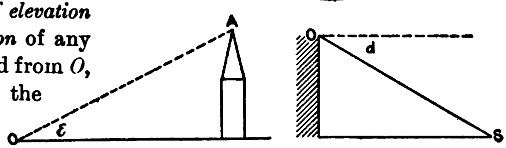
32. The figure represents an instrument used for locating the center of circular disks. Three pieces of metal are so joined that the edge AB bisects the angle formed by BD and BE, BD and BE being equal. Prove that AB passes through the center of the circle.



33. The next figure represents the same instrument applied to a much larger disk. Prove that AB passes through the center.

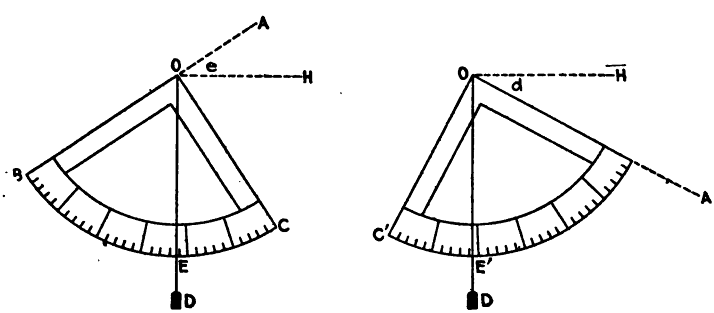


34. The *angle of elevation* or *angle of depression* of any object A as observed from O, is the angle which the line from the eye of the observer O to the object makes with a horizontal line in the same plane.



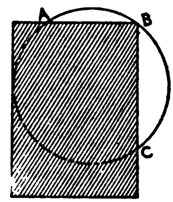
If the object lies higher than the observer, the angle is an angle of elevation, as $\angle e$. If the object lies lower than the observer, the angle is an angle of depression, as $\angle d$.

Prove that the angle of elevation of an object may be measured as follows: A quadrant is placed in vertical position so that the pro-

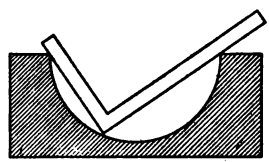


longation of BO passes through A. (This is done by means of "sights" attached at BO.) A plumb line OD, held by a pin at O, intersects the arc BC at E. The angle of elevation is measured by arc EC. Similarly arc $C'E'$ measures the angle of depression d.

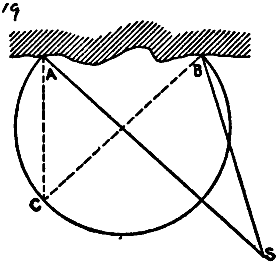
35. Prove that a diameter of a circular disk may be drawn from A as follows: Place a rectangular sheet of paper so that one vertex, B, lies in the circumference, and the adjacent side passes through A. C is the other end of the required diameter.



36. Justify the following method for testing the accuracy of a semicircular groove: Place a carpenter's square in the position indicated in the figure. If its vertex touches every point of the groove as the square slides around, the groove is a true semicircle.



37. If A and B are two lighthouses, and dangerous rocks lie within the circle ABC, while all points without the circle are navigable, prove that a ship S is not in danger of striking the rocks as long as $\angle S$ is less than $\angle C$.

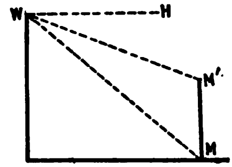


NOTE. The following six constructions (38 to 43) should be carried out by means of protractor and ruler. The required lines, or angles, should be found (approximately) by measurement.

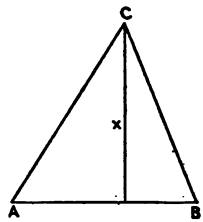
38. A vertical pole 25 ft. high casts a shadow 40 ft. long. What is the angle of elevation of the sun? (Let one inch represent 20 ft.)

39. A perpendicular cliff rises from the bank of a river. The angle of depression of the other bank, as observed from the top of the cliff, is 40°. If the cliff is 120 ft. high, what is the breadth of the river? (Let one inch represent 60 ft.)

40. At a horizontal distance of 80 ft. from the foot of a tower, the angle of elevation of its top is 50°. Find the height of the tower.

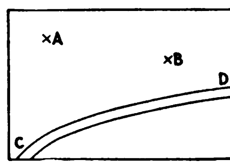


41. From a window 60 ft. above the ground the angles of depression of the top and the bottom of a monument are observed to be 20° and 40°. Find the height of the monument.

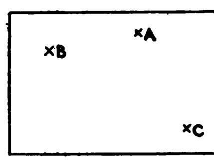


42. From two stations A and B, on opposite sides of a cloud, the angles of elevation are found to be $A = 60°$, $B = 70°$. Find the height of the cloud if $AB = 2$ mi.

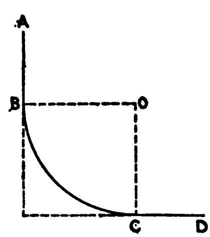
43. If the radius of the equator is 4000 mi., what is the radius of the parallel 40° N.? (Make 2000 mi. = 1 in.)



44. On a map CD represents a road and A and B two steeples. A man walking on the road from C to D measured at a certain point X, the angle $AXB = 45°$. He also noticed that X was nearer to B than to A. Locate the point X on the map.



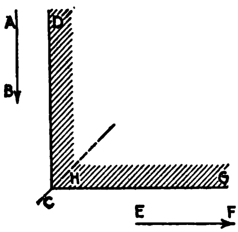
45. A, B, and C are three points on a map. At a certain point X, a surveyor measured $\angle AXB = 90°$ and $\angle AXC = 60°$. Locate the point X on the map.



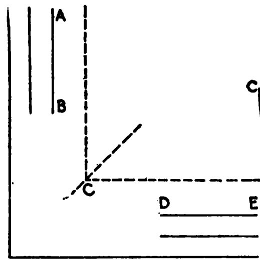
46. At the rectangular intersection of two streets the corner of the sidewalk is sometimes "rounded off." Construct the arc representing the rounded corner if its radius equals 25 ft. and if we assume the scale 1 : 500 (*i.e.* 1 in. in the drawing represents 500 in.).

47. Solve the preceding problem if the streets intersect at an acute angle.

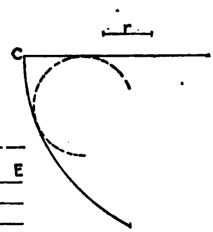
48. A wheelman riding in the direction AB ($\parallel DH$) wishes to turn the corner by moving in a circular path so that the nearest approach to the sidewalk is indicated by the point C. He continues his ride in the direction EF ($\parallel HG$). If CH bisects $\angle H$, and the distance of the first parallel lines (AB and DH) is equal to the distance of the second pair (HG and EF) construct the arc representing his path.



Ex. 48



Ex. 49

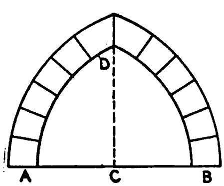


Ex. 50

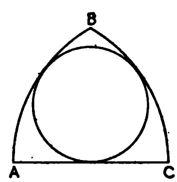
49. Solve the analogous problem for the track of a street car line.

50. "Round off" the corner C, made by a straight street and a circular one, by constructing a circle whose radius equals a given line r.

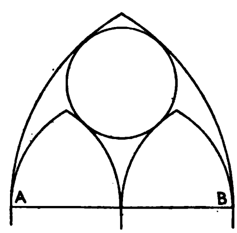
51. Construct a Gothic arch similar to the one in the diagram, if the span AB and the height CD are given. (The unknown centers of the arcs AD and BD lie in AB.)



Ex. 51



Ex. 52



Ex. 53

52. A Gothic arch ABC is equilateral, if A is the center of BC and B is the center of AC.

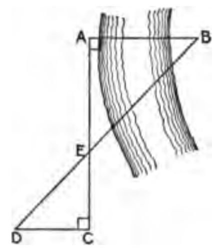
Inscribe a circle in an equilateral arch.

53. Construct the annexed figure, which consists of three equilateral arches and a circle touching the three, if AB is given.

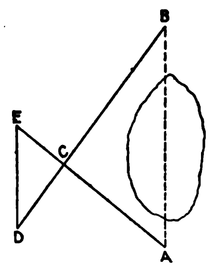
BOOK III

54. The shadow of a steeple upon level ground is 120 ft., while a pole 8 ft. long casts a shadow 3 ft. long. How high is the steeple?

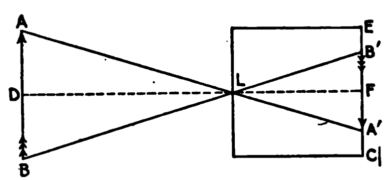
55. To determine AB, the width of a river, measure AC at right angles to AB, and CD at right angles to AC. By sighting from D to B locate the point E. What is the length of AB if $AE = 200$ ft., $EC = 25$ ft., $DC = 20$ ft.?



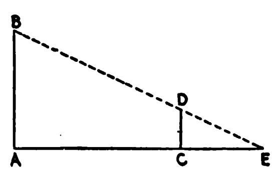
56. Show how the distance AB, which cannot be measured directly, may be determined by the following measurements: $AE = 80$ ft., $EC = 20$ ft., $DB = 100$ ft., $DC = 25$ ft., $ED = 30$ ft.



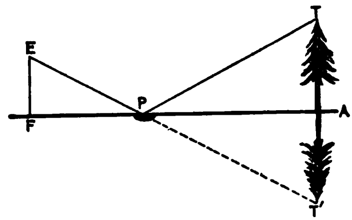
57. The location of the image A' of a point A, formed in a photographer's camera, is approximately found by drawing a straight line AA' through the center of the lens L. If CE is the position of the photographic plate, then $A'B'$ is the image of AB. How large is $A'B'$ if $AB = 6$ ft., $LD = 12$ ft., and $LF = 6$ in.?



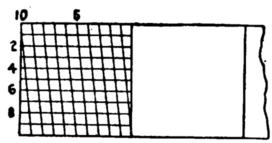
58. The top of a tower AB appears in a straight line with D, the top of a pole DC, when viewed from E. If EA is horizontal, $EC = 4$ ft., $DC = 3$ ft., and $CA = 36$ ft., what is the height of the tower?



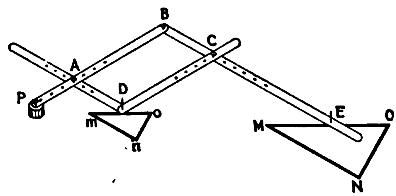
59. An observer sees the reflected image of the top of a tree formed by a pool of water in the direction ET'. If FPA is a horizontal straight line, both EF and TA are perpendicular to FPA, $EF = 5$ ft., $EP = 8$ ft., and $PA = 30$ ft.; find the height of the tree.



60. If the side of the square in the diagram ("diagonal square") is unity, show how we may measure .1, .2, .65, .34, etc.

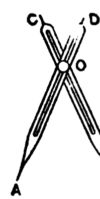


61. A *pantograph* is a machine for drawing a figure MNO similar to a given figure mno. It is used for enlarging and reducing maps and drawings. It consists of four bars, which are jointed at A, B, C, and D, and which are made parallel by making $AB = DC$ and $AD = BC$. P turns about a fixed pivot, and pencils are carried at D and E. The length of PA and CE is so adjusted that $\frac{PA}{AD} = \frac{DC}{CE}$.

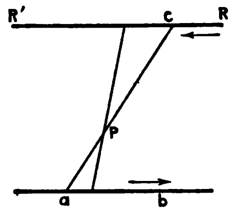


Prove that (1) P, D, and E are in a straight line, (2) $\frac{PD}{PE} = \frac{PA}{PB}$, (3) any straight line MN is parallel to mn and $\frac{MN}{mn} = \frac{PA}{PB}$, (4) any triangle MNO drawn by the pantograph is similar to the given triangle mno.

62. The figure represents an instrument (*proportional compasses*) for obtaining a line CD which is any given fraction of a given line AB. By adjusting the length of OC, OD we may make this fraction equal to $\frac{2}{5}$ of AB, etc. Prove that $CD = \frac{2}{5}$ of AB, if $OC = \frac{2}{5}\ OB$ and $OD = \frac{2}{5}\ OA$.

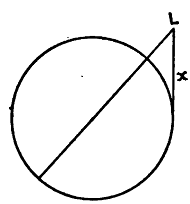


63. A map is drawn to the scale of 1 in. to 100 mi. How far apart are two places that are $3\frac{1}{2}$ in. apart on the map?



64. A man walking at the rate of 3 mi. per hour on a straight road ab which is parallel to a straight railroad RR' observes that a pole P appears in the same line of sight with a point C of a moving train. If the distance of P from ab is 10 ft. and the distance between RR' and ab is 100 ft., what is the rate of the moving train? Does it move at uniform speed?

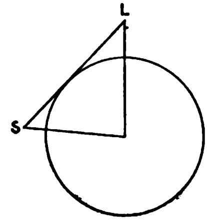
NOTE. The following examples which relate to the earth can often be simplified by bearing in mind that some numbers are only approximations. Thus, if a secant equals $8000\frac{96}{5280}$ mi., we may drop the fraction and make it equal to 8000, since the error in assuming the diameter of the earth as 8000 is much greater than the small fractions. Similarly we may measure the distance between two points on the earth's surface by a straight line instead of an arc, etc.



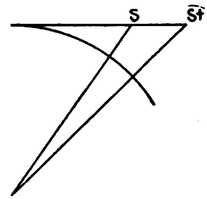
65. If the diameter of the earth is 8000 mi., how far can you see from (*a*) a lighthouse 96 ft. high? (*b*) From a mountain 2400 ft. high?

66. Prove that n, the number of miles one can see at sea from an elevation of h ft., can be found approximately by the formula $n = \sqrt{\frac{3h}{2}}$.

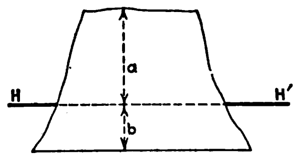
67. A swimmer places his eye at the surface of a smooth lake, and finds that the top of a sailboat 4 mi. away is just visible. How high is the top of the sailboat above the water?



68. From the deck of a steamer 24 ft. above the sea the top of a lighthouse that is 40 ft. high can just be seen beyond the horizon. Find the distance between steamer and lighthouse.



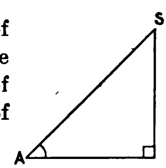
69. From the deck of a steamer which is 54 ft. above sea level the top of the mast of a sailboat appears in line with the horizon. If the sailboat is 2 mi. distant from the steamer, find the height of its mast.



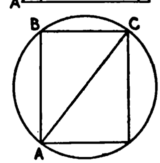
70. From the deck of a steamer which is 54 ft. above sea level an iceberg is sighted and the height of its visible part is divided by the line of the horizon HH' into two parts a and b so that $a = 3\,b$. If the distance of the iceberg from the steamer is 3 mi., find the height of the iceberg.

71. From a point 6 ft. above sea level the visible horizon has a radius of 3 mi. Find the diameter of the earth.

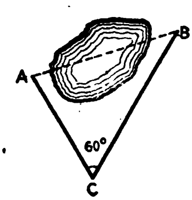
72. The Harder funicular railway near Interlaken (Switzerland) has a length of 1593 yd., and an inclination of 30°. If the lower station has an elevation of 1900 ft. above sea level, find the elevation of the upper station.



73. A skyrocket is seen to explode at an angle of elevation $A = 45°$. If the flash is seen $\frac{3}{4}$ sec. before the report of the explosion is heard, find the height of the skyrocket when it exploded. (Assume velocity of sound equal to 330 meters per second.)



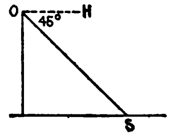
74. The strongest beam that can be cut from a given round log is the one in which the height AB is to the width BC as $\sqrt{2}:1$. Draw a diameter AC and *construct* B so that $AB : BC = \sqrt{2}:1$.



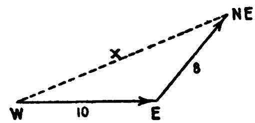
75. What is the distance between two points A and B on opposite sides of a hill if $AC = 20$ rd., $BC = 32$ rd., and $\angle C = 60°$?

76. A vertical pole on a horizontal plane casts a shadow 50 ft. long when the angle of elevation (usually called "altitude") of the sun equals 30°. Find the height of the pole.

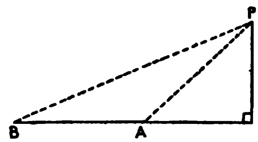
77. From a cliff 300 ft. high, the angle of depression of a ship is 45°. Find the distance of the ship from the top of the cliff.



78. A sailing vessel is propelled by the wind in an easterly direction at the rate of 10 mi. per hour. At the same time a current carries it N. E. at the rate of 8 mi. per hour. What is the true speed of the vessel per hour?



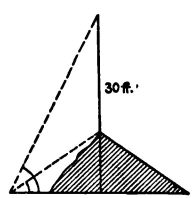
79. The angle of elevation of an inaccessible point P, measured at A, equals 45°. At a point B, 500 ft. more distant, the angle of elevation of P equals 22° 30′. If AB is a horizontal line, what is the elevation of P above A?



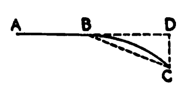
80. From a balloon which is directly above one town the angle of depression of another town is found to be 30°. Find the height of the balloon if the towns are 3 mi. apart.

81. Two observers on opposite sides of a cloud measure the angles of elevation; $A = 45°$ and $B = 67° \, 30'$. Find the height of the cloud if $AB = 3$ mi.

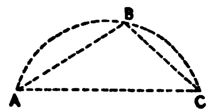
82. A flagstaff 30 ft. high stands on top of a mound, and the angles of elevation of the top and the bottom of the pole are respectively 60° and 30°. Find the height of the mound.



83. The track ABC consists of a straight line AB and the arc BC. Find the radius of arc BC if line $BC = 50$ yd., and the perpendicular CD dropped upon the prolongation of AB equals 4 yd.

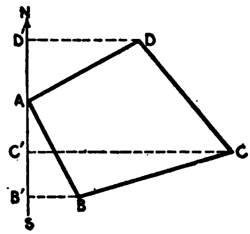


84. A circular track is to be constructed so as to pass through the points A, B, and C. If $AB = 50$ yd., $BC = 30$ yd., and $AC = 70$ yd., find the radius of the arc ABC.



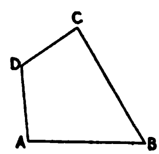
BOOK IV

85. To measure the area of a quadrangular field $ABCD$, a surveyor measures the perpendiculars drawn to a line NS and the segments made on NS as follows. $AD' = 6$ rd., $AC' = 5$ rd., $C'B' = 4$ rd., $D'D = 7$ rd., $C'C = 12$ rd., and $B'B = 4$ rd. Find the area of $ABCD$.



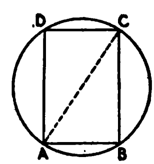
86. Determine the area of a pentagonal field $ABCDE$, if $AB = 18$, $BC = 24$, $CD = 28$, $DE = 24$, $EA = 10$, $AC = 30$, and $AD = 26$.

87. An irregular quadrangular field $ABCD$ must be divided into three equivalent parts by lines passing through A. Construct the lines of division.



88. On a map whose scale is $1:500$, how many square feet are represented by a polygon equivalent to 3 sq. in.?

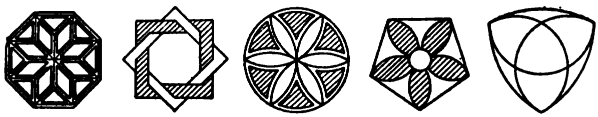
89. From a given round log a rectangular beam is to be cut so as to make its cross section $ABCD$ as large as possible. Construct $ABCD$.



90. A triangular field must be divided into 4 equivalent parts by lines parallel to one side. Construct the lines of division.

BOOK V

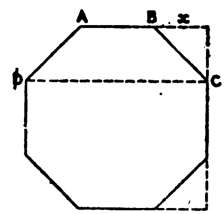
91. Construct the following patterns.



92. It is claimed that a rectangle (*e.g.* an envelope, a window, a picture, etc.) is most pleasing to the eye if its length and its width have the same ratio as the segments of a line which is divided in extreme and mean ratio. If a window is 4 ft. wide, what should be its height, if the two dimensions have the above-mentioned ratio?

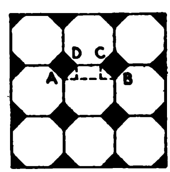
93. The side of a tower whose base is a regular hexagon measures 10 ft. Find the area of the ground occupied by the tower.

94. AB, the side of a regular octagonal tower, equals 8 ft.; find CD, the distance between two opposite walls.



95. Find the area of the ground covered by the tower described in the preceding exercise.

96. The linoleum pattern in the figure consists of regular octagons (white) and squares (black). If the side of the squares equals 2 in., what is the length of AB, the distance between two opposite sides of an octagon?

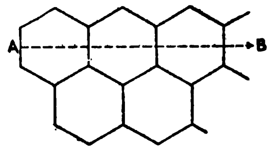


97. In the same figure find the area of an octagon ($AD = 2$ in.).

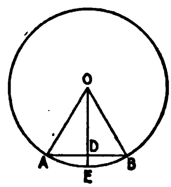
98. If the length of AB is given, construct the pattern of the preceding exercise.

99. If the length of AB (preceding figure) equals 6 in., calculate the side of a square.

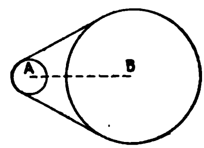
100. The floor of a room whose length is AB is covered with regular hexagonal tiles. How many tiles are in one row that extends over the entire length (AB) of the room, if each side of the hexagons equal 2 in., $AB = 10$ ft., and the tiles are placed in the position indicated in the figure?



101. The approximate value of the circumference of a circle is sometimes found by carpenters as follows. Draw the equilateral triangle AOB and produce its altitude OD to E. The circumference equals $6\,AO + DE$. Find the error if π is equal to 3.1416.



102. Two wheels A and B are connected by a belt, and their radii are respectively 2 ft. and 9 ft. If the larger wheel (B) makes 40 revolutions per minute, how many revolutions per minute will the smaller wheel make?

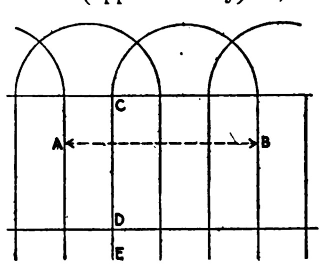


103. In the figure of the preceding exercise find the length of the belt if $AB = 14$ ft. (Radii are 2 ft. and 9 ft.)

104. The gauge of an automobile (*i.e.* the distance between two wheels on the same axis) equals 4 ft. $8\frac{1}{2}$ in. How many inches more than an inner wheel does an outer wheel travel when the car turns around a rectangular corner?

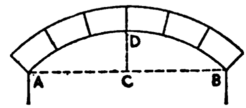
105. The circumference of the equator is (approximately) 25,000 mi. Imagine a concentric circle whose circumference is 1 ft. longer. What is the difference of the radii of the two circles?

106. The annexed figure represents a small wire fence used to protect flower beds, etc. How many feet of wire are needed per running foot of fence if $AB = 1$ ft., $CD = 9$ in., $DE = 3$ in.?



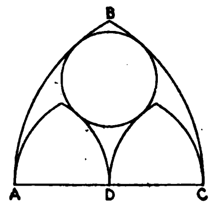
Ex. 106

107. If two streets meet at an angle of 120°, and an automobile turns from one into the other, what is the difference between the distances traveled by an outer and an inner wheel of an automobile? (Gauge 4 ft. $8\frac{1}{2}$ in.)



108. The span of a circular arch AB equals 10 ft., and its height $CD = 2$ ft. Find the radius of the arch.

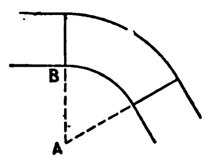
109. If the earth's orbit be assumed to be a circle whose radius = 93,000,000 mi., and the year = 365 da., how many miles per hour does the earth travel?



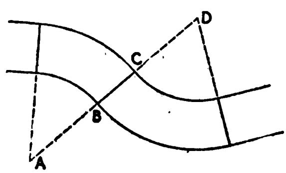
110. Find the area of an equilateral Gothic arch ABC, if $AC = 4$ ft.

111. In the same figure, what is the area of each of the smaller equilateral arches if $AD = DC = 2$ ft.?

112. In the same diagram, what is the area of the circle that touches the three equilateral arches?



113. A sidewalk 5 ft. wide turns a corner. Find the area of the curved portion of the sidewalk if the radius $AB = 8$ ft. and $\angle A = 60°$.



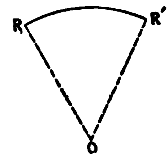
114. Determine the area of the curved portion of the road drawn in the figure, if $AB = 90$ ft., $BC = 50$ ft., $CD = 90$ ft., $\angle A = 45°$, and $\angle D = 60°$.

115. How many miles per hour does a point whose latitude equals 45° travel in consequence of the earth's rotation about its axis P if we assume the diameter of the earth equal to 8000 mi., and the time of one rotation equal to 23 hr. 56 min.?

116. It is found that a water pipe whose diameter is 2 in. supplies half the amount of water that is needed in a building. What would be the diameter of a pipe that would supply the entire required amount of water?

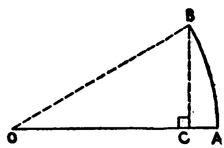
117. What should be the diameter of a water pipe that supplies the same amount of water as both an 8-in. and a 6-in. pipe, if we assume that all other conditions, as pressure, etc., are the same in the three pipes?

118. A rail 50 ft. long must be bent through what angle (*i.e.* $\angle O$ formed by perpendiculars at its end) if the radius of curvature equals 360 ft.?



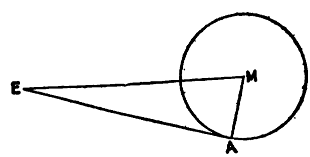
119. In laying a curved track, a rail 55 ft. long is bent through an angle of 17° 30′. Find the radius of curvature.

120. If O is the center of arc AB, $BC \perp OA$, and $\angle O$ is small, then the difference between AB and BC is very small. If we have to find the value of BC, we may, in such cases, find the length of arc AB instead. If $\angle O = 4°$, the error due to this simplification equals .00005 of the result, and for smaller angles the error is much smaller, since it is approximately proportional to the cube of the angle.



If BC is a tower standing on a horizontal plane AO, $BO = 2000$ ft. and $\angle O = 2°$. Find the height of the tower.

121. The apparent diameter of the moon equals 30′, *i.e.* a line EM drawn from earth (E) to the center of the moon (M) and a tangent EA include an angle of 15′. If $EM = 243{,}000$ mi., find the radius of the moon.



122. The apparent diameter of the sun is approximately the same as that of the moon (preceding problem). What is the diameter of the sun if its distance from the earth equals 93,000,000 mi.?